POSITIVSTELLENSÄTZE FOR ALGEBRAS OF MATRICES

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ABSTRACT. The paper is concerned with various types of noncommutative Positivstellensätze for the matrix algebra $M_n(\mathcal{A})$, where $\mathcal{A}$ is an algebra of operators acting on a unitary space, a path algebra, a cyclic algebra or a formally real field. Some new types of Positivstellensätze are proposed and proved. It is shown by examples that they occur. There are a number of results stating that a type of Positivstellensatz is valid for $M_n(\mathcal{A})$ provided that it holds for $\mathcal{A}$.

1. INTRODUCTION AND PRELIMINARIES

Positivstellensätze in real algebraic geometry express positive or non-negative polynomials on semi-algebraic sets in terms of weighted sums of squares of polynomials [PD],[Ma]. They can be considered as generalizations of E. Artin's theorem on the solution of Hilbert 17th problem. Non-commutative Positivstellensätze are fundamental results of a new emerging mathematical field that might be called non-commutative real algebraic geometry (see [S2] for a recent survey and some basic concepts). In the last decade a number of non-commutative Positivstellensätze have been obtained for various classes of ∗-algebras (see e.g. [H], [HMP],[S5], [S6], [S7], [C2]). Despite of all these results it is not even clear what a proper generalization of Artin's theorem for non-commutative ∗-algebras should be (some proposals have been made in [S2], Section 4.1).

The purpose of the present paper is threefold. Our main aim is to prove non-commutative Positivstellensätze for ∗-algebras of matrices over various classes of (commutative or non-commutative) unital ∗-algebras or ∗-fields. The corresponding Positivstellensätze will be precisely the theorems stated in the sections of the paper. Secondly, our emphasize will be on different versions of such Positivstellensätze especially concerning the involved numerator and denominators sets. In the course of this we discover a number of new types of non-commutative Positivstellensätze. All of them can be viewed as generalizations of Artin's theorem to non-commutative ∗-algebras. And finally, we want to elaborate some methods and notions that can be used to develop non-commutative Positivstellensätze. Apart from diagonalization techniques for matrices the notion of a conditional expectation will play a crucial role in this respect.

In the various sections of this paper we are concerning with different ∗-algebras of matrices and different non-commutative Positivstellensätze. Let us explain, slightly symplifying, what all our Positivstellensätze have in common. There are always "natural" notions of positivity in the corresponding algebras. Often positive elements are those elements which act as positive operators on unitary spaces or have positive point evaluations. In many cases they are defined as elements which are positive with respect to all possible ∗-orderings. It can be shown that in many cases these positivity notions are in fact equivalent. We shall denote the positive elements of a ∗-algebra $\mathcal{A}$ by $\mathcal{A}_+$. A Positivstellensatz expresses, roughly speaking, positive elements in algebraic terms built on sums of squares by allowing denominator sets.

Let us briefly explain basic types of non-commutative Positivstellensätze that will appear in this paper. All of them can be considered as possible generalizations of Artin's theorem to

general $*$-algebras. For let $\mathcal{A}$ denote a unital $*$-algebras, $\mathcal{A}^\circ$ the set of elemens sof $\mathcal{A}$ which are not zero divisors and $\mathcal{A}_+$ the set of positive elements of $\mathcal{A}$. We shall say that a

- *Positivstellensatz of type I* holds for $x \in \mathcal{A}_+$ if $x \in \sum \mathcal{A}^2$,
- *Positivstellensatz of type II* holds for $x \in \mathcal{A}_+$ if there exists a $c \in \mathcal{A}^\circ$ such that

$$c^* x c \in \sum \mathcal{A}^2,$$

- *Positivstellensatz of type III* holds for $x \in \mathcal{A}_+$ if exists an $c \in \mathcal{A}^\circ \cap \sum \mathcal{A}^2$ such that

$$xc = cx \text{ and } xc \in \sum \mathcal{A}^2,$$

- *Positivstellensatz of type IV* holds for $x \in \mathcal{A}_+$ if are pairwise commuting elements $c_1, \ldots, c_k \in \mathcal{A}^\circ \cap \sum \mathcal{A}^2$ and pairwise commuting elements $b_1, \ldots, b_n \in \sum \mathcal{A}^2$ such that

$$xc_j = c_j x \text{ for } j = 1, \ldots k \text{ and } xc_1 \ldots c_k = b_1 \ldots b_n.$$

If a Positivstellensatz is valid for all $x \in \mathcal{A}_+$, we will simply say that it holds for $\mathcal{A}$.

Probably the most general version of a Positivstellensatz is obtained if one allows general denominator sets and numerators sets as defined in Section 2. There are further natural versions by adding conditions on the sets of denominators (for instance, by requiring denominator from the center or from distinguished commutative subalgebas). Also it should be emphasized that all these versions and types essentially depend on the chosen set of positive elements $\mathcal{A}_+$.

Of course, type I is the most desirable version. There are quite a few $*$-algebras for which a Positivstellensatz of type I holds. Type I is valid for the rational functions $\mathbb{R}(x_1, \ldots, x_d)$ (by Artin's theorem), for the free polynomial algebra in d generators (by Helton's theorem), by the trigonometric polynomials in one variable (by the Riesz-Fejer theorem) and for $*$-algebra generated by the shift operator (as shown recently in [SS2]). Type I does not hold for the polynomial algebra $\mathbb{R}[x_1, \ldots, x_d]$, but type II does (again by Artin's theorem). The strict Positivstellensätze proved in [S5, S6, S7] are all of type II. We shall see in Section 10.3 below that there exist positive elements of a cyclic $*$-algebra for which a Positivstellensatz of type II is not valid, but there is a corresponding result with products of commuting squares in the denominator or likewise in the numerator. In the above terminology this means that a Positivstellensätze of types III and IV hold. This observation was in fact the starting point for our search to more general versions of Positivstellensätze.

The following simple example illustrates how the product of a positive element with commutings sums of squares becomes a sum of squares.

Example 1. Let $\mathcal{A}$ be the Weyl algebra $\mathbb{C}\langle a, a^* | aa^* - a^*a = 1 \rangle$. The $*$-algebra $\mathcal{A}$ acts on the unitary space $\mathcal{D}$ of all finite complex sequences $\varphi = (\varphi_0, \ldots, \varphi_n, 0, \ldots)$ with scalar product $\langle \varphi, \psi \rangle = \varphi_0 \overline{\psi}_0 + \varphi_1 \overline{\psi}_1 + \varphi_2 \overline{\psi}_2 + \ldots$ by

$$a\varphi = (\varphi_1, \sqrt{2}\varphi_2, \sqrt{3}\varphi_3, \ldots) \text{ and } a^*\varphi = (0, \varphi_1, \sqrt{2}\varphi_2, \sqrt{3}\varphi_3, \ldots).$$

Define $\mathcal{A}_+ = \{x \in \mathcal{A} : \langle x\varphi, \varphi \rangle \geq 0 \text{ for } \varphi \in \mathcal{D}\}$ and $N = a^*a$. Then N acts diagonal on the orthonormal basis $e_n := (\delta_{nk})$, $n \in \mathbb{N}_0$, of the unitary space $\mathcal{D}$, that is, $Ne_n = ne_n$ for $n \in \mathbb{N}_0$. Hence a polynomial $f(N) \in \mathbb{C}[N]$ is in $\mathcal{A}_+$ if and only if $f(n) \geq 0$ for all $n \in \mathbb{N}_0$.

Since $aa^* - a^*a = 1$, $c_k := (a^*)^k a^k = N(N-1) \cdots (N-(k-1)) \in \sum \mathcal{A}^2$. It can be proved (see [FS] or [SS1]) that $f(N) \in \sum \mathcal{A}^2$ if and only if there are polynomials $g_0, \ldots, g_k \in \mathbb{C}[N]$ such that

$$(1) \qquad f(N) = g_0(N)^* g_0(N) + c_1 g_1(N)^* g_1(N) + \cdots + c_k g_k(N)^* g_k(N).$$

Set $x_n = (N-n)(N-(n+1))$ for $n \in \mathbb{N}$. Then we have $x_n \in \mathcal{A}_+$ and $x_n \notin \sum \mathcal{A}^2$, because x_n is not of the form (1). But $c_n x_n = x_n c_n$ and $c_n x_n = c_{n+2} \in \sum \mathcal{A}^2$.

Let $0 < n_1 < n_2 < \cdots < n_m$. Then $x := x_{n_1} \cdots x_{n_m} \in \mathcal{A}_+$ and $c_{n_1}, \ldots, c_{n_m}$ are pairwise commuting elements of $\sum \mathcal{A}^2$ such that $xc_{n_j} = c_{n_j}x$ for $j=1, \ldots, m$ and $c_{n_1} \cdots c_{n_m}x =$

$c_{n_1+2} \cdots c_{n_m+2}$. It is not diffcult to check that the set of polynomials of the form (1) is closed under multiplication. Hence $c_{n_1} \cdots c_{n_m} x \in \mathcal{A}^2$.

Let us describe the content of this paper. Section 2 collects some basic definitions and facts which are used throughout the text. In a recent paper [Po] it was proved that a Positivstellensatz of type I is valid for path algebras. In Section 3 we apply a conditional expectation from an appropriate matrix algebra to give a simple alternative proof of this result. Section 4 deals with the matrix $*$-algebra $M_n(\mathcal{A})$ over a unital $*$-algebra $\mathcal{A}$ of operators acting on a unitary space such that $\mathcal{A}$ has no zero divisors and $\mathcal{A}^\circ := \mathcal{A} \backslash \{0\}$ is a left Ore set. By developing a non-commutative diagonalization procedure we prove that a Positivstellensatz of type II holds for $M_n(\mathcal{A})$ provided that it holds for $\mathcal{A}$. Section 5 is concerning with the cross product algebra $\mathcal{A} \times_\alpha G$ of a $*$-algebra $\mathcal{A}$ by a finite group G of $*$-automorphisms of $\mathcal{A}$. It is shown that if a Positivstellensatz of type I resp. II is valid for the matrix algebra $M_n(\mathcal{A})$, then it is valid for the cross product algebra $\mathcal{A} \times_\alpha G$ as well. Here the main technical tool is a conditional expectation from the matrix algebra $M_n(\mathcal{A})$ onto the algebra $\mathcal{A} \times_\alpha G$. In Section 6 we give a new approach to Artin's theorem for matrix algebras over commutative $*$-algebras which is based on diagonalization of matrices by means of quasi-unitary matrices. In Section 7 we use conditional expectations to describe matrix polynomials which are positive semidefinite on intervals $[a, b]$ and $[a, +\infty)$. In Section 8 we derive a Positivstellensatz of type I for the matrix algebra over formally real field equipped with the diagonal involution. Sections 9 and 10 deal with cyclic $*$-algebras. In Section 9 we prove a Positivstellensatz of type IV for cyclic $*$-algebras. It states that positive elements belong to the quadratic module generated by (certain) finite products of commuting squares. In Section 10 we elaborate a number of examples. The second example contains a positive element for which the Positivstellensatz of type II is not valid, but there is a Positivstellensatz with finite products of commuting squares in the numerator. Recently a question of Procesi and Schacher [PS] was answered in the negative in [KU]. Our third example provides another counterexample to this question. In the final Section 11 we list a number of open problems concerning non-commutative Positivstellensätze.

2. Preliminaries

Let $\mathbf{R}$ be a formally real field and $\mathbf{C} = \mathbf{R}(\sqrt{-1})$ and let $\mathbf{i} = \sqrt{-1}$ be the imaginary unit. We define an involution on $\mathbf{C}$ via $(l_1 + \mathbf{i}l_2)^* = l_1 - \mathbf{i}l_2$, $l_1, l_2 \in \mathbf{R}$, and we denote by $\mathbf{R}_+ \subseteq \mathbf{R}$ the set of positive elements of $\mathbf{R}$, or equivalently, of finite sums of squares in $\mathbf{R}$. For standard notions such as formally real fields, orderings, preorderings etc. we refer to the monographs [PD], [Ma] or [Sl].

Definition 1. An associative algebra $\mathcal{A}$ over $\mathbf{R}$ (resp. over $\mathbf{C}$) is called a $*$-*algebra* if there is a map $x \mapsto x^*$ on $\mathcal{A}$ called *involution* such that for $a, b \in \mathcal{A}, \lambda \in \mathbf{R}$ (resp. $\lambda \in \mathbf{C}$):

(i) $(\lambda a + b)^* = \lambda a^* + b^*$, (resp. $(\lambda a + b)^* = \lambda^* a^* + b^*$),
(ii) $(ab)^* = b^* a^*$,
(iii) $(a^*)^* = a$.

All $*$-algebras in this paper have an identity element denoted by $\mathbf{1}_\mathcal{A}$ or simply by $\mathbf{1}$.

Let $\mathcal{V}$ be a *unitary space*. That is, $\mathcal{V}$ is a vector space over $\mathbb{C}$ equipped with a scalar product $\langle \cdot, \cdot \rangle$ which is linear in the first variable and anti-linear in the second. Let $\mathcal{L}(\mathcal{V})$ denote the space of all linear mappings of $\mathcal{V}$ into itself.

Definition 2. An associative subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{V})$ is called an O^*-algebra if for every $A \in \mathcal{A}$ there exists $B_A \in \mathcal{A}$ such that $\langle Av, w \rangle = \langle v, B_A w \rangle$ for all $v, w \in \mathcal{V}$.

Then the element B_A is uniquely determined by A and the O^*-algebra $\mathcal{A}$ becomes a $*$-algebra with involution $A \to A^* := B_A$. Following [S1], $\mathcal{L}^+(\mathcal{V})$ denotes the largest subalgebra of $\mathcal{L}(\mathcal{V})$ which is an O^*-algebra.

Definition 3. Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$ and $\mathcal{V}$ be as above. A $*$-representation of $\mathcal{A}$ on $\mathcal{V}$ is a homomorphism $\pi : \mathcal{A} \to \mathcal{L}^+(\mathcal{V})$.

For a $*$-algebra $\mathcal{A}$ let $\mathcal{A}_h = \{a \in \mathcal{A}, \ a = a^*\}$ denote the set of all self-adjoint elements of $\mathcal{A}$.

Definition 4. Let $\mathcal{A}$ be a $*$-algebra over $\mathbf{R}$ or $\mathbf{C}$. A subset $\mathcal{C} \subseteq \mathcal{A}_h$ is a *quadratic* module if
   (i) $x + \lambda y \in \mathcal{C}$ for $x, y \in \mathcal{C}, \ \lambda \in \mathbf{R}_+$,
   (ii) $z^* x z \in Ç$ for $x \in \mathcal{C}, \ z \in \mathcal{A}$,
   (iii) $\mathbf{1}_\mathcal{A} \in \mathcal{C}$.

Let $\sum \mathcal{A}^2$ be the smallest quadratic module in $\mathcal{A}$. It consists of elements of the form $x_1^* x_1 + \cdots + x_m^* x_m$ which are called *sums of squares.*

Definition 5. A quadratic module $\mathcal{C} \subseteq \mathcal{A}_h$ is called a *non-commutative preordering* if $c_1 c_2 \in \mathcal{C}$ for all $c_1, c_2 \in \mathcal{C}, \ c_1 c_2 = c_2 c_1$.

We denote by $\sum_{nc} \mathcal{A}^2$ the smallest non-commutative preordering and call the elements of $\sum_{nc} \mathcal{A}^2$ *non-commutative sums of squares.* If $\mathcal{A}$ is a commutative $*$-algebra then $\sum \mathcal{A}^2$ coincides with $\sum_{nc} \mathcal{A}^2$, but in general we have $\sum \mathcal{A}^2 \neq \sum_{nc} \mathcal{A}^2$.

Definition 6. Let $a \in \mathcal{A}_h$. A subset $\mathcal{S}_a \subseteq \mathcal{A}_h$ is called a *denominator set* if
   (i) $a \in \mathcal{S}_a$,
   (ii) if $b \in \mathcal{S}_a$ and $x \in \mathcal{A}$, then $x^* b x \in \mathcal{S}_a$,
   (iii) if $c \in \sum_{nc} \mathcal{A}^2$ commutes with $b \in \mathcal{S}_a$, then $cb \in \mathcal{S}_a$.

The preceding definitions are motivated by the following well-known simple fact.

Lemma 1. *Suppose $\mathcal{A}$ be a (complex) $*$-algebra of all bounded operators on a (complex) Hilbert space and $\mathcal{A}_+$ be the set of positive operators in $\mathcal{A}$. Then we have $\sum_{nc} \mathcal{A}^2 \subseteq \mathcal{A}_+$. If $a \in \mathcal{A}_+$ then $\mathcal{S}_a \subseteq \mathcal{A}_+$.*

Proof. It suffices to show that $ab \in \mathcal{A}_+$ when $a, b \in \mathcal{A}_+$ and $ab = ba$. By the functional calculus for bounded self-adjoint operators (see e.g. Theorem VII.1 in [RS]), there exists a self-adjoint operator $b^{1/2}$ such that $(b^{1/2})^2 = b$ and $ab^{1/2} = b^{1/2}a$. Since $a \in \mathcal{A}_+$, we have

$$\langle ab\varphi, \varphi \rangle = \langle b^{1/2} a b^{1/2} \varphi, \varphi \rangle = \langle a b^{1/2} \varphi, b^{1/2} \varphi \rangle \geq 0$$

for $\varphi \in \mathcal{H}$, that is $ab \in \mathcal{A}_+$. $\square$

Definition 7. Let $\mathcal{A}$ be a $*$-algebra over $\mathbf{R}$ or $\mathbf{C}$. A linear map $p : \mathcal{A} \to \mathcal{B}$ is called a *conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$ if
   (i) $p(a^*) = p(a)^*$, $p(b_1 a b_2) = b_1 p(a) b_2$ for all $a \in \mathcal{A}, \ b_1, b_2 \in \mathcal{B}, \ p(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$, and
   (ii) $p(\sum \mathcal{A}^2) \subseteq \sum \mathcal{A}^2 \cap \mathcal{B}$.
A linear map p satisfying only condition (i) is called a $\mathcal{B}$-*bimodule projection* of $\mathcal{A}$ onto $\mathcal{B}$. A conditional expectation p will be called a *strong conditional expectation* if
   (ii') $p(\sum \mathcal{A}^2) \subseteq \sum \mathcal{B}^2$.

Conditional expectations for general $*$-algebras have been introduced in [SS1]. In [R] and [SS1] they are used for the study of induced $*$-representations. In this paper conditional expectations are important tools to prove Positivstellensätze for $*$-algebras.

Let $\mathbf{L}$ denote $\mathbf{R}$ or $\mathbf{C}$ and let $\mathcal{A}$ be a $*$-algebra over $\mathbf{L}$. We denote by $M_n(\mathbf{L})$ be $*$-algebra of matrices over $\mathbf{L}$ considered with the standard linear base $\{E_{ij}\}_{i,j=\overline{1,n}}$ and with multiplication and involution defined by

$$(2) \qquad E_{ij} \cdot E_{kl} = \delta_{jk} \cdot E_{il}, \ (\lambda E_{ij})^* := \lambda^* E_{ji}, \text{ for } i, j, k, l \in \{1, \ldots, n\}, \ \lambda \in \mathbf{L}.$$

We define $M_n(\mathcal{A}) := \mathcal{A} \otimes_\mathbf{L} M_n(\mathbf{L})$ with involution given by $(\sum_{ij} a_{ij} \otimes E_{ij})^* = \sum a_{ij}^* \otimes E_{ji}$. We conclude this section with an elementary lemma.

Lemma 2. *Let $\mathcal{A}$ be a $*$-algebra. Each element of $\sum M_n(\mathcal{A})^2$ is a finite sum of "rank one" squares*

$$\sum_{ij} y_i^* y_j \otimes E_{ij} = \left(\sum_i y_i \otimes E_{ki}\right)^* \left(\sum_j y_j \otimes E_{kj}\right), \ y_i \in \mathcal{A}, \ i = 1, \ldots, n,$$

where $k \in \{1, \ldots, n\}$ is fixed.

Proof. The proof follows from the computation

$$\left(\sum_{i,j=1}^n a_{ij} \otimes E_{ij}\right)^* \left(\sum_{i,j=1}^n a_{ij} \otimes E_{ij}\right) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} \otimes E_{ij}\right)^* \left(\sum_{j=1}^n a_{ij} \otimes E_{ij}\right) =$$

$$= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} \otimes E_{kj}\right)^* \left(\sum_{j=1}^n a_{ij} \otimes E_{kj}\right).$$

$\square$

3. Positivstellensatz of Type I for Path Algebras

Many important $*$-algebras are quotients of path algebras of $*$-quivers, see e.g. [CBH]. A Positivstellensatz for path algebras was recently proved by Popovych [Po]. The aim of this section to give an alternative proof of this result (Theorem 1) by using a conditional expectation.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a $*$-quiver (an equivalent notion a $*$-double of a quiver was considered in [CBH], [Po] etc.). This means that Γ is a directed multi-graph (i.e. multiple arrows between the vertices and knots are allowed) with a finite set of vertices $\Gamma_0 = \{e_1, \ldots, e_n\}$ and a finite set of arrows Γ_1. For each $e_i, e_j \in \Gamma_0$ let $\Gamma(e_i, e_j)$ denote the set of arrows from e_i to e_j. For an arrow $b \in \Gamma(e_i, e_j)$ we denote by $o(b) := e_i$ and $t(b) := e_j$ the origin and the terminal vertex of b, respectively. For each arrow $b \in \Gamma(e_i, e_j)$ there exists a unique arrow $b^* \in \Gamma(e_j, e_i)$ and we have $(b^*)^* = b$. For a knot $b \in \Gamma(e, e)$, $e \in \Gamma_0$, we assume that $b^* \neq b$.

A path in Γ is a finite sequence of arrows $b_1 b_2 \ldots b_k$ such that $t(b_i) = o(b_{i+1})$, $i = 1, \ldots, k-1$. We consider each vertex $e_i \in \Gamma_0$ as a path of the length zero. Let $\mathcal{B}$ denote the union of the set of all paths in Γ with 0. For two pathes $b_1 b_2 \ldots b_m$ and $c_1 c_2 \ldots c_l$ we define their product to be $b_1 b_2 \ldots b_m c_1 c_2 \ldots c_l$ if $t(b_m) = o(c_1)$ and 0 otherwise. Then $\mathcal{B}$ becomes a semigroup with respect to this multiplication. The semigroup algebra of this semigroup $\mathcal{B}$ with involution determined by $b \mapsto b^*$ for $b \in \Gamma_0$ is called the *path algebra* $\mathbb{C}\Gamma$.

As in [Po] we define an embedding $\epsilon : \mathbb{C}\Gamma \to M_n(\mathcal{F})$, where $\mathcal{F}$ is the free $*$-algebra with generator set Γ_1, by

$$\epsilon(b) := b \otimes E_{ij} \text{ for } b \in \Gamma(e_i, e_j) \text{ and } \epsilon(e_i) := 1 \otimes E_{ii}.$$

Our proof of Theorem 1 below uses the following slight generalization of Helton's theorem [H].

Proposition 1. *Let $\mathcal{F}_m = \mathbb{C}\langle a_1, \ldots, a_m, a_1^*, \ldots, a_m^* \rangle$ be the free $*$-algebra with m generators and let $X = X^* \in M_n(\mathcal{F}_m)$. Then $\rho(X) \geq 0$ for every finite-dimensional $*$-representation ρ of $M_n(\mathcal{F}_m)$ if and only if $X \in \sum M_n(\mathcal{F}_m)^2$.*

To prove this proposition we need the following technical result (see e.g. Lemma 2, [S3]).

Lemma 3. *Let $\mathcal{A}$ be a unital $*$-algebra which has a faithful $*$-representation π (that is, $\pi(a) = 0$ implies that $a = 0$) and is a union of a sequence of finite dimensional subspaces E_n, $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: If $a \in \sum \mathcal{A}^2$ is in E_n, then we can write a as a finite sum $\sum_j a_j^* a_j$ such that all a_j are in E_{k_n}. Then the cone $\sum \mathcal{A}^2$ is closed in $\mathcal{A}$ with respect to the finest locally convex topology on $\mathcal{A}$.*

Proof of Proposition 1. Since the if part is trivial, it suffices to prove the only if part. The main step of this proof is to show that the cone $\sum M_n(\mathcal{F}_m)^2$ is closed in the finest locally convex topology on $M_n(\mathcal{F}_m)$. For this we apply Lemma 3 to the $*$-algebra $\mathcal{A} := M_n(\mathcal{F}_m)$. The $*$-algebra $\mathcal{F}_m$ has a faithful $*$-representation (see e.g. [S3]), so has $M_n(\mathcal{F}_m)$. Let E_k be the vector space of generated by the elements $w \otimes E_{ij}$, $i, j = 1, \ldots, n$, where w runs over the monomials in $\mathcal{F}_m$ of degree $\leq k$. Let $A = \sum A_j^* A_j \in E_{2k}$. Comparing the degrees of elements in the main diagonal of A, we conclude that each A_j is in E_k. Hence the assumptions of Lemma 3 are fulfilled, so $\sum M_n(\mathcal{F}_m)^2$ is closed.

Now we proceed almost verbatim as in the proofs of Propositions 4 and 5 in [S3]. Assume to the contrary that $X \notin \sum M_n(\mathcal{F}_m)^2$. By the separation theorem for convex sets there exists a linear functional f on $M_n(\mathcal{F}_m)$ such that f is nonnegative on $\sum M_n(\mathcal{F}_m)^2$ and $f(X) < 0$. If π_f denotes the representation of $M_n(\mathcal{F}_M)$ obtained by the GNS-construction from f, there is a vector φ of the representation space V such that $f(A) = \langle \pi_f(A)\varphi, \varphi \rangle$ for all $A \in M_n(\mathcal{F}_m)$. Let P be the projection of $\mathcal{V}$ onto the finite-dimensional subspace $\pi(E_{2k})\varphi$. Since $a_1, \ldots, a_m$ are generators of the free algebra $\mathcal{F}_m$, there is a finite dimensional $*$-representation ρ of $M_n(\mathcal{F}_m)$ on $P\mathcal{V}$ defined by $\rho(a_j)v = P\pi_f(a_j)v$, $v \in P\mathcal{V}$, $j=1, \ldots, m$. By construction we have $\pi_f(A)\varphi = \rho(A)\varphi$ and hence $\langle \pi(B)\varphi, \varphi \rangle = \langle \rho(B)\varphi, \varphi \rangle$ for all $B \in E_{2k}$. In particular, $f(x) = \langle \rho(X)\varphi, \varphi \rangle < 0$ which contradicts the assumption. $\qquad\square$

Our next aim is to construct a strong conditional expectation from $M_n(\mathcal{F})$ onto $\mathbb{C}\Gamma$. For each $i, j = 1, \ldots, n$ we define a linear mapping $\mathfrak{p}_{ij} : \mathcal{F} \to \mathcal{F}$ as follows. For an element $b_1 b_2 \ldots b_k \in \mathcal{F}$, where $b_1, b_2, \ldots, b_k \in \Gamma_1$, put $\mathfrak{p}_{ij}(b_1 b_2 \ldots b_k) = b_1 b_2 \ldots b_k$ if $b_1 b_2 \ldots b_k$ is a path in Γ from e_i to e_j, that is,

$$t(b_i) = o(b_{i+1}), \ i = 1, \ldots, k-1, \ o(b_1) = e_i, \ t(b_k) = e_j.$$

Otherwise we set $\mathfrak{p}_{ij}(b_1 b_2 \ldots b_k) = 0$. Also we set $\mathfrak{p}_{ij}(\mathbf{1}) := \mathbf{1}$. We will need the following auxiliary

Lemma 4. *For all $i, j = 1, \ldots, n$ and $x, y \in \mathcal{F}$ we have:*

(i) $\mathfrak{p}_{ij}(x^*) = \mathfrak{p}_{ji}(x)^*$,
(ii) $\mathfrak{p}_{ij}(x^* y) = \sum_{k=1}^{n} \mathfrak{p}_{ki}(x)^* \mathfrak{p}_{kj}(y)$.

Proof. Both equations follow directly from the definition of $\mathfrak{p}_{ij}$. $\qquad\square$

For an arbitrary element $X = \sum_{ij} x_{ij} \otimes E_{ij}$, $x_{ij} \in \mathcal{F}$ of $M_n(\mathcal{F})$, we define

$$\mathfrak{P}(X) := \sum_{ij} \mathfrak{p}_{ij}(x_{ij}) \otimes E_{ij}.$$

Proposition 2. *The mapping $\mathfrak{P}$ is a strong conditional expectation from $M_n(\mathcal{F})$ onto $\mathbb{C}\Gamma$.*

Proof. Some easy computations show that $\mathfrak{P}$ is a $\mathbb{C}\Gamma$-bimodule projection. We prove the strong positivity property of $\mathfrak{P}$. For let $Y \in \sum M_n(\mathcal{F})^2$. By Lemma 2, Y is a finite sum of "rank one" squares, that is, we have $Y = \sum_{i,j} y_i^* y_j \otimes E_{ij}$, where $y_i \in \mathcal{F}$. Using the definition of $\mathfrak{P}$ and Lemma 4, (ii) we compute

$$\mathfrak{P}(\sum_{i,j=1}^{n} y_i^* y_j \otimes E_{ij}) = \sum_{i,j=1}^{n} \mathfrak{p}_{ij}(y_i^* y_j) \otimes E_{ij} = \sum_{i,j=1}^{n} \sum_{k=1}^{n} \mathfrak{p}_{ki}(y_i)^* \mathfrak{p}_{kj}(y_j) \otimes E_{ik} \cdot E_{kj} =$$

$$= \sum_{k=1}^{n} \sum_{i,j=1}^{n} (\mathfrak{p}_{ki}(y_i) \otimes E_{ki})^* (\mathfrak{p}_{kj}(y_j) \otimes E_{kj}) =$$

$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \mathfrak{p}_{ki}(y_i) \otimes E_{ki} \right)^* \left(\sum_{j=1}^{n} \mathfrak{p}_{kj}(y_j) \otimes E_{kj} \right) \in \sum \mathbb{C}\Gamma^2.$$

$\qquad\square$

Combining the preceding two propositions we obtain the following Positivstellensatz.

Theorem 1. *Let $X = X^* \in \mathbb{C}\Gamma$. Then $\pi(X) \geq 0$ for every finite-dimensional $*$-representation π of the path algebra $\mathbb{C}\Gamma$, if and only if $X = \sum_{j=0}^{k} X_j^* X_j$ for some elements $X_j \in \mathbb{C}\Gamma$.*

4. Artin's Theorem for Matrices over Noncomutative $*$-Algebras

Throughout this section we suppose that $\mathcal{A}$ is a unital $*$-algebra without zero divisors such that $\mathcal{A}^\circ := \mathcal{A}\setminus\{0\}$ satisfies the left Ore condition (that is, given $a \in \mathcal{A}, s \in \mathcal{A}^\circ$, there exist $b \in \mathcal{A}, t \in \mathcal{A}^\circ$ such that $ta = bs$.)

We denote by $\mathcal{D}_n(\mathcal{A})$ the diagonal matrices of $M_n(\mathcal{A})$, by $\mathcal{D}_n(\mathcal{A})^\circ$ the diagonal matrices of $M_n(\mathcal{A})$ with non-zero entries on the diagonal, and by $\mathcal{L}_n(\mathcal{A})$ the matrices $X = (x_{ij}) \in M_n(\mathcal{A})$ such that $x_{ij} = 0$ for $i \neq j$ and $x_{ii} \neq 0$ for all i.

4.1. In this first subsection we develop a general diagonalization procedure for hermitian matrices over $\mathcal{A}$. It might be of some interest in itself.

Let $a \in \mathcal{A}, \mathfrak{b} = (b_1, \ldots, b_n) \in M_{1,n}(\mathcal{A})$, $\mathfrak{b}^* := (b_1^*, \ldots, b_n^*)^t \in M_{n,1}(\mathcal{A})$ and $C = (c_{ij}) \in M_n(\mathcal{A})$ and consider a block matrix $A = A^* \in M_{n+1}(\mathcal{A})$ defined by

$$(3) \qquad A = \begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{b}^* & C \end{pmatrix}.$$

Assume that $a \neq 0$. By the left Ore property of $\mathcal{A}^\circ = \mathcal{A}\setminus\{0\}$ each right fraction $b_i^* a^{-1}$ is a left fraction. All these left fractions can be brought to a common denominator. That is, there exist elements $s \in \mathcal{A}^\circ$ and $f_1, \ldots, f_n \in \mathcal{A}$ such that $b_i^* a^{-1} = s^{-1} f_i$ or, equivalently $sb_i^* = f_i a$ for $i = 1, \ldots, n$. Set $\mathfrak{f} = (f_1, \ldots, f_n)^t \in M_{n,1}(\mathcal{A})$. Since $A = A^*$ and hence $a = a^*$, we have

$$(4) \qquad s\mathfrak{b}^* = \mathfrak{f}a \text{ and } a\mathfrak{f}^* = \mathfrak{b}s^*.$$

Let $sCs^* := (sc_{ij}s^*)$, $\mathfrak{f} = (f_1, \ldots, f_n) \in M_{n,1}(\mathcal{A})$ and $\mathfrak{f}a\mathfrak{f}^* := (f_i a f_j) \in M_n(\mathcal{A})$ and put

$$(5) \qquad D = sCs^* - \mathfrak{f}a\mathfrak{f}^*.$$

Lemma 5. *Let $x \in \mathcal{A}$ and $Y \in M_n(\mathcal{A})$. If (4) holds, then*

$$(6) \qquad \begin{pmatrix} x & 0 \\ -Y\mathfrak{f} & Ys \end{pmatrix} \begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{b}^* & C \end{pmatrix} \begin{pmatrix} x & 0 \\ -Y\mathfrak{f} & Ys \end{pmatrix}^* = \begin{pmatrix} xax^* & 0 \\ 0 & YDY^* \end{pmatrix}$$

Proof. We compute the matrix on the left-hand side by applying equation (4) several times and obtain

$$\begin{pmatrix} xa & x\mathfrak{b} \\ -Y\mathfrak{f}a + Ys\mathfrak{b}^* & -Y\mathfrak{f}\mathfrak{b} + YsC \end{pmatrix} \begin{pmatrix} x & 0 \\ -Y\mathfrak{f} & Ys \end{pmatrix}^* = \begin{pmatrix} xa & x\mathfrak{b} \\ 0 & YsC - Y\mathfrak{f}\mathfrak{b} \end{pmatrix} \begin{pmatrix} x^* & -\mathfrak{f}^*Y^* \\ 0 & s^*Y^* \end{pmatrix} =$$

$$= \begin{pmatrix} xax^* & -xa\mathfrak{f}^*Y^* + x\mathfrak{b}s^*Y^* \\ 0 & YsCs^*Y - Y\mathfrak{f}\mathfrak{b}s^*Y^* \end{pmatrix} = \begin{pmatrix} xax^* & 0 \\ 0 & YsCs^*Y^* - Y\mathfrak{f}a\mathfrak{f}^*Y^* \end{pmatrix} = \begin{pmatrix} xax^* & 0 \\ 0 & YDY^* \end{pmatrix}.$$

$\square$

Now we specialize the element $x \in \mathcal{A}$ and the matrix $Y \in M_n(\mathcal{A})$. Suppose $x \neq 0$. Applying once more the left Ore property we can write all right fractions $(Y\mathfrak{f})_i x^{-1} \equiv \sum_i y_{ij} f_j x^{-1}$, $i = 1, \ldots, n$, as left fractions with a common denominator, that is, there are elements $u \in \mathcal{A}^\circ$ and $g_1, \ldots, g_n \in \mathcal{A}$ such that $(Y\mathfrak{f})_i x^{-1} = u^{-1} g_i$ for $i = 1, \ldots, n$. Setting $\mathfrak{g} = (g_1, \ldots, g_n)^t$, we have

$$(7) \qquad uY\mathfrak{f} = \mathfrak{g}x.$$

Lemma 6. *If (4) and (7) are satisfied, then*

$$(8) \qquad \begin{pmatrix} x & 0 \\ 0 & uYs \end{pmatrix} \begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{b}^* & C \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & uYs \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ \mathfrak{g} & uI_n \end{pmatrix} \begin{pmatrix} xax^* & 0 \\ 0 & YDY^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{g} & uI_n \end{pmatrix}^*$$

Proof. Let us denote by L and R the matrices on the left and right hand-sides, respectively. We compute the right-hande side and obtain

$$R = \begin{pmatrix} xax^* & 0 \\ \mathfrak{g}xax^* & uYDY^* \end{pmatrix} \begin{pmatrix} 1 & \mathfrak{g}^* \\ 0 & u^*I_n \end{pmatrix} = \begin{pmatrix} xax^* & xax^*\mathfrak{g}^* \\ \mathfrak{g}xax^* & \mathfrak{g}xax^*\mathfrak{g}^* + uYDY^*u^* \end{pmatrix}$$

By (5) and (7), we compute

$$\mathfrak{g}xax^*\mathfrak{g}^* + uYDY^*u^* = uY\mathfrak{f}a(uY\mathfrak{f})^* + uYDY^*u^* = uY(\mathfrak{f}a\mathfrak{f}^* + D)Y^*u^* = uYsCs^*Y^*u.$$

Therefore, by (7) and (4), we continue and derive

$$R = \begin{pmatrix} xax^* & xa\mathfrak{f}^*Y^*u^* \\ uY\mathfrak{f}ax & uYsCs^*Y^*u^* \end{pmatrix} = \begin{pmatrix} xax^* & x\mathfrak{b}(uYs)^* \\ uYs\mathfrak{b}^*x^* & uYsC(uYs)^* \end{pmatrix} = L$$

which proves the assertion of the Lemma. $\square$

Remark. Retaining the preceding notations, we have

(9)
$$\begin{pmatrix} 1 & 0 \\ \mathfrak{g} & uI_n \end{pmatrix} \begin{pmatrix} x & 0 \\ -Y\mathfrak{f} & Ys \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & uYs \end{pmatrix}.$$

Lemma 7. *Let $Y \in \mathcal{L}_n(\mathcal{A})$. Then there is another matrix $T \in \mathcal{L}_n(\mathcal{A})$ such that $TY \in \mathcal{D}_n(\mathcal{A})^\circ$.*

Proof. We proceed by induction on n. Suppose that the assertion is proved for n and let $Y \in \mathcal{L}_{n+1}(\mathcal{A})$. We write

$$\begin{pmatrix} y_0 & 0 \\ \mathfrak{z} & Y_n \end{pmatrix}$$

with $y_0 \in \mathcal{A}_0^\circ, \mathfrak{z} \in M_{n,1}(\mathcal{A})$ and $Y_0 \in \mathcal{L}_n(\mathcal{A})$. By induction hypothesis there is a matrix $T_n \in \mathcal{L}_n(\mathcal{A})$ such that $T_nY_n \in \mathcal{L}_n(\mathcal{A})$. Note that $y_0 \neq 0$ by the definition of $\mathcal{L}_k(\mathcal{A})$. By the Ore property there exist elements $t_i, s_i \in \mathcal{A}^\circ$ such that $(T_n\mathfrak{z})_iy_0^{-1} = s_i^{-1}t_i$, $i = 1, \ldots, n$. Let S be the diagonal matrix with entries s_i and t the row with entries t_i. Since then $ty_0 = ST_n\mathfrak{z}$,

$$T = \begin{pmatrix} 1 & 0 \\ -\mathfrak{z} & ST_n \end{pmatrix}$$

has the desired property. $\square$

4.2.   From now on we suppose that $\mathcal{A}$ is an O^*-algebra on a unitary space $\mathcal{V}$. Then $M_n(\mathcal{A})$ is an O^*-algebra acting on $\mathcal{V}_n = \mathcal{V} \oplus \cdots \oplus \mathcal{V}$ (n times). If A is an element of $\mathcal{A}$ resp. $M_n(\mathcal{A})$ and $\mathcal{E}$ is a linear subspace of $\mathcal{V}$ resp. $\mathcal{V}_n$, we shall write $A \geq 0$ on $\mathcal{E}$ when $\langle A\varphi, \varphi \rangle \geq 0$ for all $\varphi \in \mathcal{E}$. Define

(10) $\mathcal{A}_+ = \{a \in \mathcal{A} : a \geq 0 \text{ on } \mathcal{V}\}, \quad M_n(\mathcal{A})_+ = \{A \in M_n(\mathcal{A}) : A \geq 0 \text{ on } \mathcal{V}_n\}.$

Now let A be a matrix given by (3) and retain the above notation. Let $\mathcal{E}$ and $\mathcal{F}_n$ be linear subspaces of $\mathcal{V}$ and $\mathcal{V}_n$, respectively.

Lemma 8. *$A \geq 0$ on $(\mathcal{E}, \mathcal{F}_n)$ if and only if $a \geq 0$ on $\mathcal{E}$, $C \geq 0$ on $\mathcal{F}_n$ and*

(11) $|\langle \mathfrak{b}\varphi, \varphi_1 \rangle|^2 \leq \langle a\varphi, \varphi_1 \rangle \langle C\varphi, \varphi \rangle$ *for* $\varphi_1 \in \mathcal{E}$, $\varphi \in \mathcal{F}_n$.

Proof. Let α and β be complex numbers and put $\psi_{\alpha,\beta} := (\alpha\varphi_1, \beta\varphi)$. Then we compute

(12) $\langle A\psi_{\alpha,\beta}, \psi_{\alpha,\beta} \rangle = \alpha\overline{\alpha}\langle a\varphi_1, \varphi_1 \rangle + \alpha\overline{\beta}\langle \mathfrak{b}^*\varphi, \varphi_1 \rangle + \overline{\alpha}\beta\langle \mathfrak{b}\varphi, \varphi_1 \rangle + \beta\overline{\beta}\langle C\varphi, \varphi \rangle.$

Clearly, $A \geq 0$ on $(\mathcal{E}, \mathcal{F}_n)$ if and only if $a \geq 0$ on $\mathcal{E}$, $C \geq 0$ on $\mathcal{F}_n$ and $\langle A\psi_{\alpha,\beta}, \psi_{\alpha,\beta} \rangle \geq 0$ for all $\varphi_1 \in \mathcal{E}, \varphi \in \mathcal{F}_n$ and $\alpha, \beta \in \mathbb{C}$. Since the numbers $\alpha, \beta \in \mathbb{C}$ in equation (12) are arbitrary, it follows that the latter is equivalent to the inequality (11) as stated in the Lemma. $\square$

Corollary 1. *If $A \geq 0$ on $\mathcal{V}_{n+1}$ and $a = 0$, then $\mathfrak{b} = 0$.*

Lemma 9.   (i) *If $A \geq 0$ on $\mathcal{E}_{n+1}$, then $a \geq 0$ on $\mathcal{E}$, $C \geq 0$ on $\mathcal{E}_n$ and $D \geq 0$ on $\mathcal{E}_n$.*

(ii) *If $a \geq 0$ on $\mathcal{E}$ and $D \geq 0$ on $\mathcal{E}$, then $A \geq 0$ on $(\mathfrak{f}^*\mathcal{E}, s^*\mathcal{E})$.*

Proof. (i): Suppose that $A \geq 0$. Then $a \geq 0$ on $\mathcal{E}$ and $C \geq 0$ on $\mathcal{E}_n$ by Lemma 8. Let $\varphi \in \mathcal{E}_n$. Using the identity $a\mathfrak{f}^* = b s^*$ by (4) and inequality (11) we conclude that

$$|\langle \mathfrak{f}a\mathfrak{f}^*\varphi, \varphi\rangle|^2 = |\langle \mathfrak{f}\mathfrak{b}s^*\varphi, \varphi\rangle|^2 = |\langle \mathfrak{b}s^*\varphi, \mathfrak{f}^*\varphi\rangle|^2 \leq \langle a\mathfrak{f}^*\varphi, \mathfrak{f}^*\varphi\rangle \langle Cs^*\varphi, s^*\varphi\rangle = \langle \mathfrak{f}a\mathfrak{f}^*\varphi, \varphi\rangle \langle sCs^*\varphi, \varphi\rangle.$$

If $\langle \mathfrak{f}a\mathfrak{f}^*\varphi, \varphi\rangle \neq 0$, then we have $\langle \mathfrak{f}a\mathfrak{f}^*\varphi, \varphi\rangle \leq \langle sCs^*\varphi, \varphi\rangle$ and so $\langle D\varphi, \varphi\rangle \geq 0$. If $\langle \mathfrak{f}a\mathfrak{f}^*\varphi, \varphi\rangle = 0$, then $\langle D\varphi, \varphi\rangle = \langle Cs^*\varphi, s^*\varphi\rangle \geq 0$, because $C \geq 0$. Thus, $D \geq 0$ on $\mathcal{E}_n$.

(ii): Let $\varphi, \psi \in \mathcal{E}_n$. Since $a \geq 0$ on $\mathcal{E}$, we have $\langle D\psi, \psi\rangle = \langle Cs^*\psi, s^*\psi\rangle - \langle \mathfrak{f}a\mathfrak{f}^*\psi, \psi\rangle \geq 0$ and hence $\langle Cs^*\psi, s^*\psi\rangle \geq \langle a\mathfrak{f}^*\psi, \mathfrak{f}^*\psi\rangle = \langle \mathfrak{b}s^*\psi, \mathfrak{f}^*\psi\rangle \geq 0$, so we obtain $C \geq 0$ on $s^*\mathcal{E}_n$ and

$$|\langle \mathfrak{b}s^*\varphi, \mathfrak{f}^*\psi\rangle|^2 = |\langle a\mathfrak{f}^*\varphi, \mathfrak{f}^*\psi\rangle|^2 \leq \langle a\mathfrak{f}^*\varphi, \mathfrak{f}^*\varphi\rangle \langle a\mathfrak{f}^*\psi, \mathfrak{f}^*\psi\rangle \leq \langle a\mathfrak{f}^*\varphi, \mathfrak{f}^*\varphi\rangle \langle Cs^*\psi, s^*\psi\rangle.$$

Therefore, $A \geq 0$ on $(\mathfrak{f}^*\mathcal{E}, s^*\mathcal{E})$ by Lemma 8. $\square$

Proposition 3. *For each matrix $A \in M_n(\mathcal{A})_+$ there exist matrices $X_+, X_- \in \mathcal{L}_n(\mathcal{A})$ such that $X_+AX_+^* \in \mathcal{D}_n(\mathcal{A})_+$ and $X_-X_+ \in \mathcal{D}_n(\mathcal{A})^\circ$.*

Proof. Let $z_1, \ldots, z_n$ be given elements of $\mathcal{A}^\circ$. In view of the subsequent application given below we prove the stronger assertion that the diagonal matrix $\mathcal{D} = X_+AX_+^*$ can be chosen the form $d_i = z_i a_i z_i^*$ for some $a_i \in \mathcal{A}_+$.

The proof is given by induction on n. Obviously, the assertion is true for $n = 1$. Let $A \in M_{n+1}(\mathcal{A})_+$. We write A in the form (3). Since $A \geq 0$, we have $C \geq 0$ by Lemma 9, so the induction hypothesis applies to the matrix C.

If $a = 0$, then $\mathfrak{b} = 0$ by Corollary 1 and it suffices to enlarge the corresponding matrices for C by putting 1 in the left upper corner and 0 elsewhere.

From now on suppose that $a \neq 0$. By the induction hypothesis, there are matrices $Y_+, Y_- \in \mathcal{L}_n(\mathcal{A})$ for which $Y_+Y_- \in \mathcal{D}_n(\mathcal{A})^\circ$ and $D_n := Y_+CY_+^* \in \mathcal{D}_n(\mathcal{A})_+$ has diagonal entries $d_i = z_i a_i z_i^*$ with $a_i \in \mathcal{A}_+$, $i = 2, \ldots, n+1$. We apply Lemma 9 with $x = z_1$ and $Y = Y_+$. Putting

$$X_+ = \begin{pmatrix} z_1 & 0 \\ -Y_+\mathfrak{f} & Y_+ s \end{pmatrix}$$

we therefore have $X_+ \in \mathcal{L}_{n+1}(\mathcal{A})$ and $X_+AX_+ \in \mathcal{D}_{n+1}(\mathcal{A})_+$ has the diagonal $z_1 a z_1^*, z_i a_i z_i^*$ for $i = 2, \ldots, n+1$. Note that $a \in \mathcal{A}_+$.

From Lemma 8 there is a matrix $T \in \mathcal{L}_n(\mathcal{A})$ such that $T \cdot (uY_- s) \in \mathcal{D}_n(\mathcal{A})^\circ$. Set

$$X_- = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{g} & uI_n \end{pmatrix}.$$

From equation (7) it follows that

$$X_-X_+ = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathfrak{g} & uI_n \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ -Y_+\mathfrak{f} & Y_+ s \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & TuYs \end{pmatrix} \in \mathcal{D}_n(\mathcal{A})^\circ.$$

$\square$

4.3. Let $A \in M_n(\mathcal{A})_+$. Then, by Proposition 3 there are diagonal matrices $D_0 \in \mathcal{D}_n(\mathcal{A})_+$, $D \in \mathcal{D}_n(\mathcal{A})^\circ$ and matrices $X_+, X_- \in \mathcal{L}_n(\mathcal{A})$ such that

(13) $$D_0 = X_+AX_+^* \quad \text{and} \quad DAD^* = X_-D_0X_-^*.$$

We shall use this result to show that a Positivstellensatz of type II holds for the matrices over $\mathcal{A}$ provided that it holds for $\mathcal{A}$ itself. More precicely, we have the following Positivstellensatz. Recall that $\mathcal{A}_+$ and $M_n(\mathcal{A})_+$ have been defined by (10).

Theorem 2. *Let $\mathcal{A}$ be an O^*-algebra. Suppose that $\mathcal{A}$ has no zero divisors and $\mathcal{A}$ satisfies the left Ore condition. Assume that for each element $a \in \mathcal{A}_+$ there exists $z \in \mathcal{A}^\circ$ such that $zaz^* \in \mathcal{A}^2$.*

Then for each matrix $A \in M_n(\mathcal{A})_+$, $n \in \mathbb{N}$, there are matrices $D_0 \in \sum \mathcal{D}_n(\mathcal{A})^2$, $D \in \mathcal{D}_n(\mathcal{A})^\circ$ and $X_+, X_- \in \mathcal{L}_n(\mathcal{A})$ such that $X_+ A X_+^* = D_0$ and $DAD^* = X_- D_0 X_-^* \in \sum M_n(\mathcal{A})^2$.

Proof. Let $z_1, \ldots, z_n$ be fixed elements of $\mathcal{A}_0$. In the above proof of the Proposition 3 it was shown that there exist elements $a_1, \ldots, a_n$ such that $X_+ A X_+^* \in \mathcal{D}_n(\mathcal{A})_+$ has the diagonal entries $z_i a z_i^*$. By the assumption we can choose $z_i \in \mathcal{A}$ such that $z_i a_i z_i^* \in \sum \mathcal{A}^2$. Then the assertion follows from the proof of Proposition 3, see also (13). $\qquad \square$

Remark. In Subsection 4.3 of [S2] a related result was obtained for matrices over the *commutative* polynomial algebra $\mathbb{R}[x_1, \ldots, x_d]$. In this result we had matrices $X_+, X_- \in \mathcal{L}_n(\mathcal{A})$ for which both products $X_+ X_-$ and $X_- X_+$ are central. In the above theorem for the *noncommutative* O^*-algebras $\mathcal{A}$ we have only the weaker assertion stating that $X_- X_+ \in \mathcal{D}_n(\mathcal{A})^\circ$.

5. POSITIVSTELLENSÄTZE FOR CROSSED PRODUCT ALGEBRAS

Let $\mathcal{A}$ be a unital $*$-algebra and let G be a finite group of $*$-automorphisms of $\mathcal{A}$. Let $\alpha_g \in \mathrm{Aut}\, G$ denote the $*$-automorphism corresponding to $g \in G$. In this section we will show how Positivstellensätze for the matrix algebra $M_n(\mathcal{A})$, $n = |G|$, can be used to derive Positivstellensätze for the crossed product algebra $\mathcal{A} \times_\alpha G$ of $\mathcal{A}$ with G.

First let us recall the defintion of the crossed product $*$-algebra $\mathcal{A} \times_\alpha G$. As a linear space it is the tensor product $\mathcal{A} \otimes \mathbb{C}[G]$ or equivalently the vector space of $\mathcal{A}$-valued functions on G with finite support. Product and involution on $\mathcal{A}$ are determined by

$$(a \otimes g)(b \otimes h) = a\alpha_g(b) \otimes gh \ \text{ and } \ (a \otimes g)^* = \alpha_{g^{-1}}(a^*) \otimes g^{-1},$$

respectively. If we identify b with $b \otimes e$ and g with $1 \otimes g$, then the $*$-algebra $\mathcal{A} \times_\alpha G$ can be considered as the universal $*$-algebra generated by the two $*$-subalgebras $\mathcal{A}$ and $\mathbb{C}[G]$ with cross commutation relations $gb = \alpha_g(b)g$ for $b \in \mathcal{A}$ and $g \in G$.

5.1. Our first aim is to construct an embedding $\mathcal{A} \times_\alpha G \hookrightarrow \mathcal{A} \otimes M_n(\mathbb{C})$. Define the linear mapping ϵ from $\mathcal{A} \times_\alpha G$ to $M_n(\mathcal{A})$ as follows:

$$(14) \qquad\qquad \epsilon : a \otimes g \mapsto \sum_{h \in G} \alpha_h(a) \otimes E_{h, hg}.$$

Lemma 10. *The mapping ϵ is an injective $*$-homomorphism of $\mathcal{A} \times_\alpha G$ to $M_n(\mathcal{A})$.*

Proof. Take $a \otimes g, \ b \otimes k \in \mathcal{A} \times_\alpha G$. Then we have

$$\epsilon(a \otimes g)\epsilon(b \otimes k) = \left(\sum_{h \in G} \alpha_h(a) \otimes E_{h, hg} \right) \left(\sum_{l \in G} \alpha_l(b) \otimes E_{l, lk} \right) =$$

$$= \sum_{h,l \in G} \delta_{hg,l} \cdot \alpha_h(a)\alpha_l(b) \otimes E_{h, lk} = \sum_{h \in G} \alpha_h(a)\alpha_{hg}(b) \otimes E_{h, hgk} = \sum_{h \in G} \alpha_h(a\alpha_g(b)) \otimes E_{h, hgk} =$$

$$= \epsilon(a\alpha_g(b) \otimes gk) = \epsilon((a \otimes g)(b \otimes k)).$$

Analogously one checks that $(\epsilon(a \otimes g))^* = \epsilon((a \otimes g)^*)$. Thus ϵ is a $*$-homomorphism. It is easily seen that ϵ is injective. $\qquad \square$

From now on we consider $\mathcal{A} \times_\alpha G$ as a $*$-subalgebra of $M_n(\mathcal{A})$ via the embedding ϵ.

5.2. Next we define a projection $\mathfrak{P}$ from $M_n(\mathcal{A})$ onto $\mathcal{A} \times_\alpha G$. For every $g \in G$ let β_g denote the linear mapping of $M_n(\mathcal{A})$ onto itself defined by

$$\beta_g : a \otimes E_{m,k} \mapsto \alpha_g(a) \otimes E_{gm,gk}, \ a \in \mathcal{A}, \ m, k, g \in G. \tag{15}$$

Lemma 11. *The map $g \mapsto \beta_g$ is a well-defined action of G on $M_n(\mathcal{A})$ by $*$-automorphisms.*

Proof. The proof is given by straightforward computations. We omit the detail. □

Define $\mathfrak{P}$ as the average over the action $g \mapsto \beta_g$, that is,

$$\mathfrak{P}(a \otimes E_{m,k}) := \frac{1}{n} \sum_{g \in G} \beta_g(a \otimes E_{m,k}) = \frac{1}{n} \sum_{g \in G} \alpha_g(a) \otimes E_{gm,gk}. \tag{16}$$

Proposition 4. *The mapping $\mathfrak{P}$ is a faithful strong conditional expectation from $M_n(\mathcal{A})$ onto $\mathcal{A} \times_\alpha G$.*

Proof. It follows from the formulas (14) and (15) that $\mathcal{A} \times_\alpha G$ is the stable $*$-subalgebra under the action of G on $M_n(\mathcal{A})$. Being an average over an action of a finite group $\mathfrak{P}$ is a faithful conditional expectation by [SS1], Proposition 5.

We prove the strong positivity property of $\mathfrak{P}$. Take an element $\sum_{m,k \in G} c_{m,k} \otimes E_{m,k} \in M_n(\mathcal{A})$. Then we have

$$\mathfrak{P}\left(\left(\sum_{m,k \in G} c_{m,k} \otimes E_{m,k}\right)^* \left(\sum_{m,k \in G} c_{m,k} \otimes E_{m,k}\right)\right) = \mathfrak{P}\left(\sum_{m,k,l \in G} c_{m,k}^* c_{m,l} \otimes E_{k,l}\right) =$$

$$= \frac{1}{n} \sum_{g \in G} \beta_g\left(\sum_{m,k,l \in G} c_{m,k}^* c_{m,l} \otimes E_{k,l}\right) = \frac{1}{n} \sum_{g \in G} \sum_{m,k,l \in G} \alpha_g(c_{m,k}^* c_{m,l}) \otimes E_{gk,gl} = \tag{17}$$

$$= \frac{1}{n} \sum_{m \in G} \left(\sum_{g,k \in G} \alpha_g(c_{m,k}) \otimes E_{gm,gk}\right)^* \left(\sum_{g,l \in G} \alpha_g(c_{m,l}) \otimes E_{gm,gl}\right)$$

The embedding formula (14) implies that

$$\sum_{g,k \in G} \alpha_g(c_{m,k}) \otimes E_{gm,gk} = \sum_{k \in G} \sum_{g \in G} \alpha_{gm}(\alpha_{m^{-1}}(c_{m,k})) \otimes E_{gm,gm(m^{-1}k)} =$$

$$= \sum_{k \in G} \sum_{g \in G} \alpha_g(\alpha_{m^{-1}}(c_{m,k})) \otimes E_{g,gm^{-1}k} = \sum_{k \in G} \alpha_{m^{-1}}(c_{m,k}) \otimes m^{-1}k.$$

Using the latter equation we proceed in (17) and derive

$$= \frac{1}{n} \sum_{m \in G} \left(\sum_{k \in G} \alpha_{m^{-1}}(c_{m,k}) \otimes m^{-1}k\right)^* \left(\sum_{k \in G} \alpha_{m^{-1}}(c_{m,k}) \otimes m^{-1}k\right) \in \sum(\mathcal{A} \times_\alpha G)^2.$$

□

5.3. As in Section 4, we suppose that $\mathcal{A}$ is an O^*-algebra acting on a unitary space $\mathcal{V}$.

Let us recall the definition of the regular covariant representation of $\mathcal{A} \times_\alpha G$. The representation space $\mathcal{V}^{|G|}$ is a direct sum $\oplus_{g \in G} \mathcal{V}$ of $|G|$ copies of $\mathcal{V}$. For $e \in \mathcal{V}$ and $k \in G$ we denote by e_k the element of $\oplus_{g \in G} \mathcal{V}$ which has e at the place k and is 0 otherwise. Let $g \in G$ and $a \in \mathcal{A}$. We define linear mappings $\rho(g)$, $\pi(a)$ and $\pi_{creg}(a \otimes g)$ on the vector space $\mathcal{V}^{|G|}$ by

$$\rho(g)e_k := e_{kg^{-1}}, \quad \pi(a)e_k := (\alpha_k(a)e)_k, \quad \pi_{creg}(a \otimes g)e_k = \pi(a)\rho(g)e_k, \quad e \in \mathcal{V}, \ k \in G.$$

Some simple computations show that π_{creg} is a well-defined $*$-representation of the $*$-algebra $\mathcal{A} \times_\alpha G$ on $\mathcal{V}^{|G|}$ and that,

$$\pi(\alpha_g(a)) = \rho(g)\pi(a)\rho(g)^*, \ a \in \mathcal{A}, \ g \in G.$$

The $*$-representation π_{creg} (or likewise the triple $(\pi, \rho, \mathcal{V}^{|G|})$) are called the *regular covariant $*$-representation* of the crossed product algebra $\mathcal{A} \times_\alpha G$. Define

$$(\mathcal{A} \times_\alpha G)_+ = \{x = x^* \in \mathcal{A} \times_\alpha G : \pi_{creg}(x) \geq 0\}.$$

Using the embedding formula (14) one easily verifies that the action of $\mathcal{A} \times_\alpha G \subseteq M_n(\mathcal{A})$ coincides with π_{creg}, thus proving the

Lemma 12. $(\mathcal{A} \times_\alpha G)_+ \subseteq M_n(\mathcal{A})_+$, where $M_n(\mathcal{A})_+$ is defined by (10).

Theorem 3. *If the Positivstellensatz of type I is valid for $M_n(\mathcal{A})$, then it holds also for $\mathcal{A} \times_\alpha G$.*

Proof. Let $x = x^* \in (\mathcal{A} \times_\alpha G)_+$. Then we have $x \in M_n(\mathcal{A})_+$ by the preceding lemma and hence $x \in \sum M_n(\mathcal{A})^2$ by the Positivstellensatz for $M_n(\mathcal{A})_+$. Since $\mathfrak{P}$ is a strong conditional expectation by Proposition 4, we obtain $\mathfrak{P}(x) = x \in \sum(\mathcal{A} \times_\alpha G)^2$. $\square$

Theorem 4. *If the Positivstellensatz of type III holds for $M_n(\mathcal{A})$, it holds for the cross product algebra $\mathcal{A} \times_\alpha G$ as well.*

Proof. Suppose that $x = x^* \in (\mathcal{A} \times_\alpha G)_+$. Then $x \in M_n(\mathcal{A})_+$. Since the Positivstellensatz of type III holds for $M_n(\mathcal{A})$, there exists an element $y \in M_n(\mathcal{A})^\circ \cap \sum M_n(\mathcal{A})^2$ such that $xy = yx$ and $xy \in \sum M_n(\mathcal{A})^2$. Because $\mathfrak{P}$ is a strong conditional expectation by Proposition 4, we have $x\mathfrak{P}(y) = \mathfrak{P}(xy) = \mathfrak{P}(yx) = \mathfrak{P}(y)x \in \sum(\mathcal{A} \times_\alpha G)^2$ and $\mathfrak{P}(y) \in \sum(\mathcal{A} \times_\alpha G)^2$.

It is left to show that $\mathfrak{P}(y) \in (\mathcal{A} \times_\alpha G)^\circ$. Moreover, we show that $\mathfrak{P}(y) \in M_n(\mathcal{A})^\circ$. Indeed, by assumption $y = \sum_i y_i^* y_i$. Let $z \in M_n(\mathcal{A})$ be such that $\mathfrak{P}(y)z = 0$. Then using defintion of $\mathfrak{P}$ we have also

$$0 = z^*\mathfrak{P}(y)z = \frac{1}{n}\sum_i \sum_{g \in G} z^*\beta_g(y_i)^*\beta_g(y_i)z = \frac{1}{n}\sum_i \sum_{g \in G}(\beta_g(y_i)z)^*(\beta_g(y_i)z).$$

Since $M_n(\mathcal{A})$ as well as $\mathcal{A}$ is an O^*-algebra, all $\beta_g(y_i)z$ are 0, hence $yz = \sum_i y_i^*(y_i z) = 0$. Since y is not a zero divizor we get $z = 0$. $\square$

Now we turn to the Positivstellensatz of type II for $\mathcal{A} \times_\alpha G$.

Theorem 5. *Let $\mathcal{A}$ be an O^*-algebra without zero divisors such that $\mathcal{A}^\circ$ is a left Ore set. Assume that the Positivstellensatz of type II holds for $\mathcal{A}$. Then the Positivstellensatz of type II holds also for the crossed-product algebra $\mathcal{A} \times_\alpha G$.*

Proof. Suppose that $X \in (\mathcal{A} \times_\alpha G)_+$. Then $X \in M_n(\mathcal{A})_+$ by Lemma 12. Therefore, by Theorem 2 there exists a diagonal matrix $Y \in \mathcal{D}_n(\mathcal{A})^\circ$ such that $Y^*XY \in \sum M_n(\mathcal{A})^2$. Let $Y = \sum_{i=1}^n y_i \otimes E_{g_i,g_i}$, where $y_1, y_2, \ldots, y_n \in \mathcal{A}^\circ$. Because $\alpha_{g_i^{-1}}(y_i) \in \mathcal{A}^\circ$, it follows from a repeated application of the Ore condition that there exist $z_1, z_2, \ldots, z_n \in \mathcal{A}^\circ$ such that

$$(18) \qquad \alpha_{g_1^{-1}}(y_1)z_1 = \cdots = \alpha_{g_n^{-1}}(y_n)z_n \in \mathcal{A}^\circ.$$

Put $Z = \sum_{i=1}^n \alpha_{g_i}(z_i) \otimes E_{g_i,g_i}$. We claim that $YZ \in (\mathcal{A} \times_\alpha G)^\circ$. It is easily seen that $YZ \in M_n(\mathcal{A})^\circ$. We check that YZ is invariant under β_g, $g \in G$, which implies $YZ \in \mathcal{A} \times_\alpha G$. Indeed, for a fixed $g \in G$ we compute using (18)

$$\beta_g(YZ) = \beta_g\left(\sum_{i=1}^n y_i\alpha_{g_i}(z_i) \otimes E_{g_i,g_i}\right) = \sum_{i=1}^n \alpha_g(y_i\alpha_{g_i}(z_i)) \otimes E_{gg_i,gg_i} =$$

$$= \sum_{g_i \in G} \alpha_{gg_i}(\alpha_{g_i^{-1}}(y_i)z_i) \otimes E_{gg_i,gg_i} = \sum_{g_j \in G} \alpha_{g_j}(\alpha_{g_j^{-1}}(y_j)z_j) \otimes E_{g_j,g_j} = \sum_{j=1}^n y_j\alpha_{g_j}(z_j) \otimes E_{g_j,g_j} = YZ.$$

Since $(YZ)^*XYZ = Z^*Y^*XYZ \in \sum M_n(\mathcal{A})^2$ and $X, YZ \in (\mathcal{A} \times_\alpha G)^\circ$, we obtain $\mathfrak{P}((YZ)^*XYZ) = (YZ)^*XYZ \in \sum(\mathcal{A} \times_\alpha G)^2$. $\square$

Example 2. Let us describe ϵ more explicitly in the case when $G = \{0, 1, \ldots, n-1\}$ is a cyclic group of order n generated by a $*$-automorphism σ, $\sigma^n = \text{Id}$. Then ϵ maps the element $\sum_{k=0}^{n-1} a_k \otimes k$ onto the matrix

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \sigma(a_{n-1}) & \sigma(a_0) & \cdots & \sigma(a_{n-2}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{(n-1)}(a_1) & \sigma^{(n-1)}(a_2) & \cdots & \sigma^{(n-1)}(a_0) \end{pmatrix}$$

The algebra of matrices of this form might be of interest in itself even if σ is the identity automorphism.

6. Artins Theorem for Matrices over Commutative $*$-Algebras: Quasi-Unitary Matrices

In this section $\mathcal{A}$ is a finitely generated unital commutative $*$-algebra without zero divisors over the field $\mathbb{L}$, where $\mathbb{L}$ is $\mathbb{R}$ or $\mathbb{C}$. Let $\widehat{\mathcal{A}}$ denote the set of characters on $\mathcal{A}$, that is, $\widehat{\mathcal{A}}$ is the set of all nontrivial $*$-homomorphisms $\chi : \mathcal{A} \to \mathbb{L}$. We assume that $\widehat{\mathcal{A}}$ separates the elements of $\mathcal{A}$. the latter implies in particular that $\sum_{j=1}^{n} a_j^* a_j = 0$ always implies that $a_1 = \cdots = a_n = 0$. Define

$$\mathcal{A}_+ = \{a \in \mathcal{A} : \chi(a) \geq 0 \text{ for } \chi \in \widehat{\mathcal{A}}\}, \quad M_n(\mathcal{A})_+ = \{A = (a_{ij}) \in M_n(\mathcal{A}) : (\chi(a_{ij})) \geq 0 \text{ for } \chi \in \widehat{\mathcal{A}}\}.$$

Artin's theorem for the matrix algebra $M_n(\mathbb{R}[t_1, \ldots, t_d])$ was proved independently in [GR] and in [PS]. A constructive proof based on Schur complements was first developed in [S3]. The aim of this short section is to give a proof of Artin's theorem for the matrix $*$-algebra $M_n(\mathcal{A})$ by using quasi-unitary matrices.

A matrix $T \in M_n(\mathcal{A})$ is called *quasi-unitary* if there exist an $s \in \sum \mathcal{A}^2$ such that $T^*T = TT^* = sI$. Obviously, the element s is uniquely determined by T. It will be denoted by $s(T)$.

Example 3. If $a_1, a_2 \in \mathcal{A}$, then

$$(19) \qquad\qquad T = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix}$$

is a quasi-unitary matrix and $s(T) = a_1^* a_1 + a_2^* a_2$.

To explain the approach in a simple special case we consider a 2×2-matrix

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

of $M_2(\mathcal{A})_+$ such that $a \neq 0$. Then $a \in \mathcal{A}_+$ and $ac - b^*b \in \mathcal{A}_+$. Let us assume that there exist elements $a_1, a_2, d \in \mathcal{A}$ such that $a = a_1^* a_1 + a_2^* a_2$ and $ac - b^*b = d^*d$.

Setting $(y_1, y_2)^t = T(b, d)^t$, that is, $y_1 = a_1 b - a_2^* d, y_2 = a_2 b + a_1^* d$, we have

$$a^2 A = \begin{pmatrix} aa_1 & y_1 \\ aa_2 & y_2 \end{pmatrix}^* \begin{pmatrix} aa_1 & y_1 \\ aa_2 & y_2 \end{pmatrix}.$$

Proposition 5. *Let $k \in \mathbb{N}$ and let $a_1, \ldots, a_{2^k} \in \mathcal{A}$ be given. There exists a quasi-unitary matrix $T = (t_{ij}) \in M_{2^k}(\mathcal{A})$ and a nonzero element $s \in \sum \mathcal{A}^2$ such that $t_{i1} = sa_i$ for $i = 1, \ldots, 2^k$.*

Proof. We proceed by induction on k. In the case $k = 1$ we take the matrix T given by (19). Suppose that the assertion is valid for k. If all elements $a_{2^k+1}, \ldots, a_{2^{k+1}}$ are zero, the assertion holds by the induction hypthesis. Suppose now that not all of these elements are zero. Then there are quasi-unitary matrices $A = (a_{ij}), B = (b_{ij}) \in M_{2^k}(\mathcal{A})$ and nonzero elements $s_1, s_2 \in \sum \mathcal{A}^2$ such that $a_{i1} = s_1 a_i$ and $b_{ij} = s_2 a_{2^k+i}$ for $i = 1, \ldots, 2^k$. Since not all

elements $a_{2^k+1}, \ldots, a_{2^{k+1}}$ are zero, we have $s(B) = s_2^2 \sum_{i=1}^{2^k} a_{2^k+i}^* a_{2^k+i} \neq 0$. A straightforward computation shows that the block matrix

$$(20) \qquad T = \begin{pmatrix} s_2 s(B)A & -s_1 s(B)B^* \\ s_1 s(B)B & s_2 BA^* B^* \end{pmatrix}$$

is quasi-unitary and we have $s := s_1 s_2 s(B) \in \sum \mathcal{A}^2$, $s \neq 0$ and $t_{i1} = s a_i$ for $i = 1, \ldots, 2^{k+1}$. $\qquad \square$

Artin's theorem for the $*$-algebra $M_n(\mathcal{A})$ is the following result.

Theorem 6. *Retain the assumptions and the notation stated above. For any $A \in M_n(\mathcal{A})_+$ there exists an element $c \in \mathcal{A}^\circ$ such that $c^* c A \in \sum M_n(\mathcal{A})^2$.*

Proof. Assume that the assertion holds for $n \in \mathbb{N}$. Let $A \in M_{n+1}(\mathcal{A})_+$. We write A as a block matrix

$$(21) \qquad\qquad\qquad A = \begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{b}^* & C \end{pmatrix},$$

where $a \in \mathcal{A}, \mathfrak{b} \in M_{1,n}(\mathcal{A})$, and $C \in M_n(\mathcal{A})$. If $a = 0$, the determinants of all 2×2 principal submatrices containing a are in $\mathcal{A}_+$ (see e.g. [Zh], p. 161) which in turn implies that $\mathfrak{b} = 0$. (This can be also derived from Corollary 1.) Then the assertion follows a by applying the induction hypothesis to C. From now on we suppose that $a \neq 0$.

Since $A \in M_n(\mathcal{A})_+$, we have $a \in \mathcal{A}_+$ and $D := aC - \mathfrak{b}^* \mathfrak{b} \in M_n(\mathcal{A})_+$ (see e.g. [Zh], p. 184). We assumed that $\mathcal{A}$ is a finitely generated unital commutative$*$-algebra such that $\widehat{\mathcal{A}}$ separates the elements of $\mathcal{A}$. Therefore, Artin's theorem holds for $\mathcal{A}$ which also gives the first induction step. By this theorem and by the induction hypothesis there exist elements $c_1, c_2 \in \mathcal{A}^\circ$ such that $c_1^* c_1 a \in \sum \mathcal{A}^2$ and $c_2^* c_2 D \in \sum M_n(\mathcal{A})^2$. Setting $c = c_1 c_2$ and adding zeros in the sums of squares if needed, there are $m \in \mathbb{N}$, elements $a_1, \ldots, a_{2^m} \in \mathcal{A}$ and column matrices $\mathfrak{a}_2, \ldots, \mathfrak{a}_{2^m} \in M_{1,n}(\mathcal{A})$ such that $c^* c a = \sum_{j=1}^{2^m} a_j^* a_j$ and $c^* c D = \sum_{j=2}^{2^m} \mathfrak{a}_j^* \mathfrak{a}_j$. Setting $\mathfrak{a}_1 := c\mathfrak{b}$, we obtain $c^* c a C = \sum_{j=1}^{2^m} \mathfrak{a}_j^* \mathfrak{a}_j$.

By Proposition 5 there exist an $r \in \sum \mathcal{A}^2$ and a quasi-unitary matrix $T = (t_{ij}) \in M_{2^m}(\mathcal{A})$ such that $t_{i1} = r a_i$ for $i = 1, \ldots, 2^m$. Since T is quasi-unitary, we then have

$$s(T) = r^2 \sum_{j=1}^{2^m} a_j^* a_j = r^2 c^* c a.$$

Since r, c and a are nonzero elements and $\mathcal{A}$ has no zero divisors, $s(T) \neq 0$. Putting $\mathfrak{y}_j = \sum_{k=1}^m r t_{jk} \mathfrak{a}_k$ for $j = 1, \ldots, 2^m$, we compute

$$\sum_j \mathfrak{y}_j^* \mathfrak{y}_j = \sum_j \sum_{k,l} r^2 \mathfrak{a}_k t_{jk}^* t_{jl} \mathfrak{a}_l = \sum_{k,l} r^2 \mathfrak{a}_k (T^* T)_{kl} \mathfrak{a}_l = r^2 s(T) \sum_k \mathfrak{a}_k^* \mathfrak{a}_k = r^2 s(T) c^* c a C = s(T)^2 C,$$

$$\sum_j (rcaa_j)^* \mathfrak{y}_j = \sum_{j,k} r^2 c^* a t_{j1}^* t_{jk} \mathfrak{a}_k = r^2 c^* a \sum_k (T^* T)_{1,k} \mathfrak{a}_k = r^2 c^* a s(T) \mathfrak{a}_1 = r^2 c^* c a s(T) \mathfrak{b} = s(T)^2 \mathfrak{b},$$

$$\sum_j \mathfrak{y}_j^* rcaa_j = \left(\sum_j (rcaa_j)^* \mathfrak{y}_j \right)^* = s(T)^2 \mathfrak{b}^*, \quad \sum_j (rcaa_j)^* rcaa_j = r^2 c^* c a^2 \sum_j a_j^* a_j = s(T)^2 a.$$

Setting $\mathfrak{x}_j := (rcaa_j, \mathfrak{y}_j) \in M_{1,n+1}(\mathcal{A})$, the preceding four equations mean that

$$\sum_{j=1}^{2^m} \mathfrak{x}_j^* \mathfrak{x}_j = \sum_{j=1}^{2^m} \begin{pmatrix} (rcaa_j)^* & 0 \\ \mathfrak{y}_j^* & 0 \end{pmatrix} \begin{pmatrix} rcaa_j & \mathfrak{y}_j \\ 0 & 0 \end{pmatrix} = s(T)^2 \begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{b}^* & C \end{pmatrix} = s(T)^2 A.$$

$\square$

Remark 1. The notion of quasi-unitaries used above might be useful for *non-commutative* ∗-algebras as well. Let $\mathcal{B}$ be a (not necessarily commutative) unital ∗-algebra with center $\mathcal{Z}(\mathcal{B})$. Suppose that $bz = 0$ for $b \in \mathcal{B}$ and $z \in \mathcal{Z}(\mathcal{B})$ always implies that $b = 0$ or $z = 0$. An element $b \in \mathcal{B}$ is called a *quasi-unitary* if there exists an element $s(b) \in \sum \mathcal{Z}(\mathcal{B})^2$ such that $b^*b = bb^* = s(b)$. Clearly, if a and b are quasi-unitary, so is ab and $s(ab) = s(a)s(b)$.

7. MATRIX POLYNOMIALS ON INTERVALS

In this section we give another application of conditional expectations. Let $\mathbb{C}[\mathbb{T}]$ be the algebra of trigonometric polynomials in one variable with complex coefficients. The classical Fejér-Riesz theorem says that every positive element $f \in \mathbb{C}[\mathbb{T}]$ is equal to $g\overline{g}$ for some $g \in \mathbb{C}[\mathbb{T}]$. The following non-commutative version of this theorem follows from Theorem 7 in [Ro].

Proposition 6. *A self-adjoint element* $X \in M_n(\mathbb{C}[\mathbb{T}])$ *is positive semidefinite in every point* $t \in \mathbb{T}$ *if and only if* $X = Y^*Y$ *for some* $Y \in M_n(\mathbb{C}[\mathbb{T}])$.

Let us identify the ∗-algebra $\mathbb{C}[\mathbb{T}]$ with the quotient ∗-algebra $\mathbb{C}[x,y]/\langle 1-x^2-y^2\rangle$. Since each element of $\mathbb{C}[x,y]/\langle 1-x^2-y^2\rangle$ can be written uniquely in the form $f_1 + yf_2$ with $f_1, f_2 \in \mathbb{C}[x]$, under this identification $\mathbb{C}[\mathbb{T}]$ becomes the vector space $\mathbb{C}[x] + y\mathbb{C}[x]$ with multiplication rule

$$(f_1 + yf_2)(g_1 + yg_2) = f_1f_2 + (1-x^2)f_2g_2 + y(f_1g_2 + f_2g_1), \ f_i, g_i \in \mathbb{C}[x].$$

We consider $\mathbb{C}[x]$ as a ∗-subalgebra of $\mathbb{C}[\mathbb{T}]$ and define a mapping $\mathfrak{p} : \mathbb{C}[\mathbb{T}] \to \mathbb{C}[x]$ by $\mathfrak{p}(f_1(x) + yf_2(x)) := f_1(x)$. In the same manner we consider $M_n(\mathbb{C}[x])$ as a ∗-subalgebra of $M_n(\mathbb{C}[\mathbb{T}])$ and define a projection $\mathfrak{P} : M_n(\mathbb{C}[\mathbb{T}]) \to M_n(\mathbb{C}[x])$ by $\mathfrak{P}((f_{ij})) := (\mathfrak{p}(f_{ij})), \ f_{ij} \in \mathbb{C}[\mathbb{T}]$.

Proposition 7. *The mapping* $\mathfrak{P}$ *is a conditional expectation such that*

$$\mathfrak{P}(X^*X) = Y_1^*Y_1 + (1-x^2)Y_2^*Y_2, \ X \in \mathbb{C}[\mathbb{T}], \ Y_1, Y_2 \in M_n(\mathbb{C}[\mathbb{T}]).$$

Proof. We write X as $X_1 + yX_2, \ X_1, X_2 \in M_n(\mathbb{C}[x])$. It is easily seen that $\mathfrak{P}(X^*X) = X_1^*X_1 + y^2X_2^*X_2 = X_1^*X_1 + (1-x^2)X_2^*X_2$. One can readily check that $\mathfrak{P}$ fulfils the other axioms of a conditional expectation. $\square$

Theorem 7. *Let* $F(x) \in M_n(\mathbb{C}[x])$ *be a self-adjoint matrix polynomial. Then* $F(x)$ *is positive semidefinite in every point* $x \in [a,b], a < b$ *if and only if* $F = G_1^*G_1 + (b-x)(x-a)G_2^*G_2$ *for some* $G_1, G_2 \in M_n(\mathbb{C}[x])$.

Proof. It suffices to prove the only if direction. Upon applying a linear transformation we can assume that $[a,b] = [-1,1]$. Let $F(x)$ be positive on $[-1,1]$. Then $F(x)$ is a positive element of $M_n(\mathbb{C}[\mathbb{T}])$, so that $F = G^*G$ for some $G \in M_n(\mathbb{C}[\mathbb{T}])$ by Proposition 6. From Proposition 7 we obtain $F = \mathfrak{P}(F) = \mathfrak{P}(G^*G) = G_1^*G_1 + (1-x)(x+1)G_2^*G_2$ for some $G_1, G_2 \in M_n(\mathbb{C}[x])$. $\square$

The same methods can be used to derive a Positivstellensatz for matrix polynomials on $[a,+\infty)$. First we recall a well-known result (see e.g. [Dj]).

Proposition 8. *A self-adjoint matrix polynomial* $X \in M_n(\mathbb{C}[x])$ *is positive semidefinite for all* $x \in \mathbb{R}$ *if and only if* $X = Y^*Y$ *for some* $Y \in M_n(\mathbb{C}[x])$.

Proceeding in the same way as above, we write $g \in \mathbb{C}[x]$ uniquely as $g_1 + xg_2, \ g_1, g_2 \in \mathbb{C}[x^2]$ and define a conditional expectation $\mathfrak{p}_2 : \mathbb{C}[x] \to \mathbb{C}[x^2]$ by $\mathfrak{p}_2(g) := g_1$. We then obtain

Theorem 8. *Let* $F(x) \in M_n(\mathbb{C}[x])$ *be a self-adjoint matrix polynomial. Then* $F(x)$ *is positive semidefinite for all* $x \in [a,+\infty)$ *if and only if* $F = G_1^*G_1 + (x-a)G_2^*G_2$ *for some* $G_1, G_2 \in M_n(\mathbb{C}[x])$.

Remark. For $F(x) = (F_{ij}(x)) \in M_n(\mathbb{C}[x])$ we define $\deg F := \max_{ij} \deg F_{ij}$. A closer look at the proofs of Propositions 6 and 8 allows us to estimate the degrees of G_1 and G_2 in Theorems 7 and 8. In Theorem 7 we can achieve $\deg G_1 \le \deg F, \ \deg G_2 \le \deg F - 1$ and in Theorem 8 we can have $\deg G_1 \le \frac{1}{2}\deg F, \deg G_2 \le \frac{1}{2}(\deg F - 1)$, where we mean $G_2 = 0$ when $\deg G_2 = -1$.

8. Positivstellensätze for matrices over fields

In this section $\mathbf{R}$ is a formally real field and recall that $\mathbf{R}_+$ denotes the set of finite sums of squares in $\mathbf{R}$. Let $\mathbf{C} = \mathbf{R}(\sqrt{-1}) = \mathbf{R} + i\mathbf{R}$, where $\mathbf{i} = \sqrt{-1}$, with involution defined by $(l_1 + \mathbf{i}l_2)^* = l_1 - \mathbf{i}l_2$, $l_1, l_2 \in \mathbf{R}$. The purpose of this section is to study positivity in $M_n(\mathbf{R})$ and $M_n(\mathbf{C})$ and to prove the Positivstellensatz stated as Theorem 9 below.

In what follows the symbol $\mathbf{L}$ denotes one of the fields $\mathbf{R}$ or $\mathbf{C}$ meaning that a statement or definition holds for both $\mathbf{R}$ and $\mathbf{C}$. We consider $M_n(\mathbf{L})$ as $*$-algebra with involution $(a_{ij})^* = (a_{ji}^*)$. We denote by $\mathfrak{p}_{kk} : M_n(\mathbf{L}) \to \mathbf{L}$ the mapping $(a_{ij}) \mapsto a_{kk}$ and by $\mathrm{tr} : M_n(\mathbf{L}) \to \mathbf{L}$ the mapping $\mathrm{tr} = \frac{1}{n}(\mathfrak{p}_{11} + \cdots + \mathfrak{p}_{nn})$.

Let $B = \mathrm{diag}\,(1, \lambda_1, \ldots, \lambda_{n-1}) \in M_n(\mathbf{R})$ be a diagonal matrix, where $\lambda_1, \ldots, \lambda_{n-1} \in \mathbf{R}^\circ$, and τ the involution on $M_n(\mathbf{L})$ associated with B, that is, $X^\tau := B^{-1}X^*B$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbf{L}^n$ and $\langle x, y \rangle_1 := \langle Bx, y \rangle$ the inner product on $\mathbf{L}^n$ defined by B. Then τ is the involution associated to $\langle \cdot, \cdot \rangle_1$. We first record a simple lemma.

Lemma 13. *Let p be an ordering on $\mathbf{R}$. Then the following are equivalent:*

  (i) *for fixed $k \in \{1, \ldots, n\}$ and for all $X \in M_n(\mathbf{L})$, p contains $\mathfrak{p}_{kk}(X^\tau X)$,*
  (ii) *for all $k \in \{1, \ldots, n\}$ and for all $X \in M_n(\mathbf{L})$, p contains $\mathfrak{p}_{kk}(X^\tau X)$,*
  (iii) *for all $X \in M_n(\mathbf{L})$, p contains $\mathrm{tr}(X^\tau X)$,*
  (iv) *for all $\mathfrak{f} \in \mathbf{L}^n$, p contains $\langle \mathfrak{f}, \mathfrak{f} \rangle_1$.*
  (v) *p contains $\lambda_1, \ldots, \lambda_{n-1}$.*

Proof. Equivalence (i)⇔(ii)⇔(iii)⇔(iv) follows from simple computations. Let $\mathfrak{e}_1, \ldots \mathfrak{e}_n$ be the standard base of $\mathbf{L}^n$ and let $\mathfrak{f} = \sum_{i=1}^n f_i \mathfrak{e}_i$. Then

$$\langle \mathfrak{f}, \mathfrak{f} \rangle_1 = \sum_{i=1}^n f_i^* f_i \langle \mathfrak{e}_i, \mathfrak{e}_i \rangle_1 = \sum_{i=1}^n \lambda_{i-1} f_i^* f_i, \text{ where } \lambda_0 = 1$$

which implies (iv)⇔(v). □

Definition 8. An ordering p of $\mathbf{R}$ is called $*$-*ordering* if one of the statements (i)-(v) in Lemma 13 is satisfied.

To avoid degenerate situations, we assume that $*$-orderings exist. We denote by $P_B(\mathbf{L})$ the preordering generated by $\lambda_1, \ldots, \lambda_{n-1}$. The existence of $*$-orderings is equivalent to the requirement that $P_B(\mathbf{L})$ is proper, that is, $-1 \notin P_B(\mathbf{L})$. Since a proper preordering is the intersection of all orderings containing it, $P_B(\mathbf{L})$ is the intersection of all $*$-orderings. An element $l \in \mathbf{L}$ is positive in all $*$-orderings if and only if l belongs to $P_B(\mathbf{L})$.

Lemma 14. *For $X = X^\tau \in M_n(\mathbf{L})$ the following are equivalent:*

  (i) *for a fixed $k \in \{1, \ldots, n\}$ and every $Y \in M_n(\mathbf{L})$, we have $\mathfrak{p}_{kk}(Y^\tau X Y) \in P_B(\mathbf{L})$,*
  (ii) *for every $k \in \{1, \ldots, n\}$ and every $Y \in M_n(\mathbf{L})$, we have $\mathfrak{p}_{kk}(Y^\tau X Y) \in P_B(\mathbf{L})$,*
  (iii) *for every $Y \in M_n(\mathbf{L})$, we have $\mathrm{tr}(Y^\tau X Y) \in P_B(\mathbf{L})$,*
  (iv) *for every $\mathfrak{f} = (f_1, \ldots, f_n) \in \mathbf{L}^n$, we have $\langle X\mathfrak{f}, \mathfrak{f} \rangle_1 \in P_B(\mathbf{L})$,*
  (v) *all of the principal minors of X belong to $P_B(\mathbf{L})$.*

Proof. The equivalence of the statements (i)-(iv) is proved by simple computations. The equivalence of (iv) and (v) is a slight generalization of Sylvester's criterion for positive semi-definite matrices. □

Definition 9. Let $(M_n(\mathbf{L}), \tau)_+$ be the set of all $X = X^\tau \in M_n(\mathbf{L})$ for which one of the statements (i)-(v) in Lemma 14 is satisfied. Such elements are called positive (with respect to τ).

The following Lemma contains some elementary properties of $(M_n(\mathbf{L}), \tau)_+$.

Lemma 15.   (i) $\sum X_i^\tau X_i \in (M_n(\mathbf{L}), \tau)_+$ *for arbitrary $X_i \in M_n(\mathbf{L})$.*

(ii) *If $Y = Y^\tau \in (M_n(\mathbf{L}), \tau)_+$ then $X^\tau Y X \in (M_n(\mathbf{L}), \tau)_+$ for every $X \in M_n(\mathbf{L})$.*

The following proposition seems to be missing in the literature even for the case when B is an identity matrix and $\mathbf{L} = \mathbf{R}$.

Proposition 9. *If $X, Y \in (M_n(\mathbf{L}), \tau)_+$ and $XY = YX$, then $XY \in (M_n(\mathbf{L}), \tau)_+$. In particular, $(M_n(\mathbf{L}), \tau)_+$ is a non-commutative preordering.*

Proof. We give the proof for the case $\mathbf{L} = \mathbf{R}$. The case $\mathbf{L} = \mathbf{C}$ is treated similarly. Let us fix a $*$-ordering p on $\mathbf{R}$ and let $\overline{\mathbf{R}}$ be the real closure of $(\mathbf{R}, p)$. It follows from Lemma 14 (v) that $X, Y \in (M_n(\overline{\mathbf{R}}), \tau)_+$ and it is enough to show that $XY \in (M_n(\overline{\mathbf{R}}), \tau)_+$. Since $\overline{\mathbf{R}}$ is real closed and p contains $\lambda_1, \ldots, \lambda_{n-1}$, the matrix $B^{1/2} := \mathrm{diag}\left(1, \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{n-1}}\right)$ belongs to $M_n(\overline{\mathbf{R}})$. Then the mapping $\phi(X) = B^{1/2} X B^{-1/2}$ defines a $*$-isomorphism of $(M_n(\overline{\mathbf{R}}), \tau)$ and $M_n(\overline{\mathbf{R}})$ with respect to the transpose involution. The elements $X_1 := \phi(X)$ and $Y_1 := \phi(Y)$ belong to $M_n(\overline{\mathbf{R}})_+$ and commute. It is enough to check that $X_1 Y_1 \in M_n(\overline{\mathbf{R}})_+$. This is proved by simultaneous diagonalization of X_1 and Y_1. Since X_1 is symmetric, there exists an orthogonal matrix $U \in M_n(\overline{\mathbf{R}})$ such that $U^T X_1 U$ is a diagonal matrix $\mathrm{diag}\,(x_1, \ldots, x_n) \in M_n(\overline{\mathbf{R}})$. Assume that the x_i are pairwise different (the other case is treated by choosing U more specifically). Then $U^T Y_1 U$ commutes with $U^T X_1 U$ and since the x_i are pairwise different, $U^T Y_1 U$ is also diagonal. Since U is orthognoal, both $U^T X_1 U$ and $U^T Y_1 U$ belong to $M_n(\overline{\mathbf{R}})_+$. Hence their product $U^T X_1 Y_1 U$ also belongs to $M_n(\overline{\mathbf{R}})_+$ and $X_1 Y_1 \in M_n(\overline{\mathbf{R}})_+$. $\qquad\square$

Example 4. It is proved in [PS] that for $n = 2$ we have $(M_n(\mathbf{R}), \tau)_+ = \sum (M_n(\mathbf{R}), \tau)^2$ for arbitrary $\mathbf{R}$. For $n = 3$ there is the following counterexample given in [KU]. Let $\mathbf{R} = \mathbb{R}(s, t)$ be the field of rational functions in two variables and $B = \mathrm{diag}\,(s, t, st)$. Then $\mathrm{diag}\,(st, st, st)$ is positive, but it is not a sum of squares in $(M_n(\mathbf{R}), \tau)$, see Theorem 3.2 in [KU].

Let $P_B^n(\mathbf{L}) \subseteq M_n(\mathbf{R})$ denote the set of all diagonal matrices with entries from $P_B(\mathbf{L})$.

Lemma 16. *Let $X \in M_n(\mathbf{R})$ be a diagonal matrix. Then $X \in (M_n(\mathbf{R}), \tau)_+$ if and only if $X \in P_B^n(\mathbf{L})$.*

Proof. This follows immediately from Lemma 14(iv). $\qquad\square$

The next proposition is a standard fact from linear algebra in the case $\mathbf{L} = \mathbb{R}$ and $B = I$. Its proof in the present case is completely analogous.

Proposition 10. *Let $X = X^\tau \in M_n(\mathbf{L})$. Then there exists an invertible matrix $Y = (y_{ij}) \in M_n(\mathbf{L})$ such that $Y^\tau X Y$ is diagonal.*

By Example 4, $P_B^n(\mathbf{L})$ contains elements which are not sums of squares. We prove that $P_B^n(\mathbf{L})$ generates $(M_n(\mathbf{R}), \tau)_+$ as a quadratic module.

Proposition 11. *Let $X = X^\tau \in M_n(\mathbf{L})$. Then $X \in (M_n(\mathbf{L}), \tau)_+$ if and only if there exist $Y \in M_n(\mathbf{L})$ and $D \in P_B^n(\mathbf{L})$ such that $X = Y^\tau D Y$.*

Proof. First let $X = Y^\tau D Y$ and $D \in P_B^n(\mathbf{L})$. Then $\langle Xv, v \rangle_1 = \langle Y^\tau D Y v, v \rangle_1 = \langle DY v, Y v \rangle_1 \geq 0$ in all $*$-orderings.

Conversely, suppose that $X \in (M_n(\mathbf{L}), \tau)_+$. By Proposition 10 there exists an invertible matrix $Z \in M_n(\mathbf{R})$ such that $D := ZXZ^\tau$ is diagonal. Since $X \in (M_n(\mathbf{R}), \tau)_+$ we also have $ZXZ^\tau \in (M_n(\mathbf{R}), \tau)_+$. Since D is diagonal, Lemma 16 implies that $D \in P_B^n(\mathbf{L})$. Therefore, $X = Z^{-1} D (Z^\tau)^{-1} = Z^{-1} D (Z^{-1})^\tau$. Put $Y = (Z^{-1})^\tau$. $\qquad\square$

Proposition 12. *Every element of $P_B^n(\mathbf{L}) \subseteq (M_n(\mathbf{R}), \tau)_+$ is a non-commutative sum of squares in $(M_n(\mathbf{R}), \tau)$. In the case $n = 2$ we have $P_B^n(\mathbf{L}) \subseteq \sum (M_n(\mathbf{R})\tau)^2$.*

Proof. It is enough to prove the assertion for the matrices $\lambda_m \cdot E_{kk} \in P_B(\mathbf{L})$. For $k > 1$ we compute

$$(\lambda_{k-1}E_{m+1,k})^\tau(\lambda_{k-1}E_{m+1,k}) = \lambda_{k-1}^2 B^{-1} E_{k,m+1} B E_{m+1,k} = \lambda_m \lambda_{k-1} E_{kk},$$

and we also have $E_{1k}^\tau E_{1k} = \lambda_{k-1}^{-1} E_{kk}$. Thus, $\lambda_m \cdot E_{kk}$ is the product of two commuting squares $(\lambda_{k-1}E_{m+1,k})^\tau(\lambda_{k-1}E_{m+1,k})$ and $E_{1k}^\tau E_{1k}$. In the case $k = 1$, $\lambda_m \cdot E_{kk} = (\lambda_m E_{m+1,1})^\tau(\lambda_m E_{m+1,1})$.
For $n = 2$ the inclusion $P_B^n(\mathbf{L}) \subseteq \sum(M_n(\mathbf{R})\tau)^2$ follows from $\lambda_1 E_{11}, \lambda_1 E_{22} \in \sum(M_n(\mathbf{R})\tau)^2$. $\qquad\square$

Theorem 9. *Let $X = X^\tau \in M_n(\mathbf{L})$. Then $X \in (M_n(\mathbf{L}), \tau)_+$ if and only if it is a non-commutative sum of squares. For $n = 2$ we have the equality $(M_n(\mathbf{R}), \tau)_+ = \sum(M_n(\mathbf{R})\tau)^2$.*

Proof. The assertion follows from Propositions 9, 11 and 12. $\qquad\square$

Remarks. 1. The condition $-1 \notin P_B(\mathbf{L})$ seems to be the natural definition for $(M_n(\mathbf{R}), \tau)$ of being formally real. It implies that an equality $\sum_i X_i^\tau X_i = 0$ is only possible when all $X_i = 0$.

2. The case of a non-trivial involution on the basic field can be treated more generally. Then one defines $\mathbf{C}$ as $\mathbf{R}(\sqrt{-s})$ for some $s \in \sum \mathbf{R}^2$, $s \neq 0$, that is, one takes $\mathbf{j} = \sqrt{-s}$ as imaginary unit. Since $\mathbf{R}$ is formally real, $\mathbf{C}$ is a proper extension of $\mathbf{R}$ and it is not formally real. A natural involution on $\mathbf{C}$ is defined by $(l_1 + \mathbf{j}l_2)^* = l_1 - \mathbf{j}l_2$. A field $\mathbf{C}$ arising in this way might be called "formally complex." The involution on $\mathbf{C}$ has the property that sums of "hermitian squares" $l_i^* l_i$ are contained in $\sum \mathbf{R}^2$. All definitions and statements of this section carry over to this case.

9. POSITIVSTELLENSÄTZE FOR CYCLIC ALGEBRAS

Let $\mathbf{R}, \mathbf{C}$ and $\mathbf{L}$ be as in the preceding section. Suppose that $\mathbf{L}/\mathbf{K}$ is a Galois extension with group $\mathbb{Z}/n$, σ is an automorphism of $\mathbf{L}$ which generates $\mathbb{Z}/n$, and $\mathfrak{A}$ is a cyclic algebra associated to $\mathbf{L}/\mathbf{K}$. Our aim in this section is to develop a Positivstellensatz (Theorem 10) for the cyclic algebra $\mathfrak{A}$.

Since $\mathfrak{A}$ is a cyclic algebra, there exist fixed elements $e \in \mathfrak{A}$ and $a \in \mathbf{K}^\circ$ such that

$$(22) \qquad \mathfrak{A} = \mathbf{1} \cdot \mathbf{L} \oplus e \cdot \mathbf{L} \oplus \dots e^{n-1} \cdot \mathbf{L}, \ e^n = a \cdot \mathbf{1}, \text{ and } \lambda \cdot e = e \cdot \sigma(\lambda), \text{ for } \lambda \in \mathbf{L}.$$

From (22) it follows that $\mathfrak{A} = (\mathbf{L}/\mathbf{K}, \sigma, a)$ is a $\mathbb{Z}/n$-graded algebra with standard grading $\mathfrak{A}_k = e^k \cdot \mathbf{L}$. We assume in addition that $\mathfrak{A}$ is a $\mathbb{Z}/n$-graded $*$-algebra, that is, the involution of $\mathfrak{A}$ satisfies

$$(23) \qquad\qquad\qquad \mathfrak{A}_k^* = \mathfrak{A}_{-k} \text{ for } k \in \mathbb{Z}/n.$$

By (23), $\mathfrak{A}_0 \simeq \mathbf{L}$ is invariant under the involution, where the involution of $\mathbf{L}$ is defined as in the previous section. It follows from (23) that $e^* \in \mathfrak{A}_{n-1}$, so there exists an element $l_0 \in \mathbf{L}^\circ$ such that $e^* = l_0 \cdot e^{n-1}$. Hence $e^*e = l_0 \cdot e^n = l_0 \cdot a$.

Lemma 17. *σ is a $*$-automorphism of $\mathbf{L}$.*

Proof. Applying the involution to the equality $e \cdot \sigma(l) = l \cdot e$ and replacing e^* by $l_0 \cdot e^{n-1}$ we obtain:

$$\sigma(l)^* l_0 \cdot e^{n-1} = l_0 \cdot e^{n-1} \cdot l^* = l_0 \sigma(l^*) \cdot e^{n-1} = \sigma(l^*) l_0 \cdot e^{n-1}.$$

This implies that $\sigma(l)^* = \sigma(l^*)$. $\qquad\square$

Let ϵ be the left regular representation of $\mathfrak{A}$. We identify $\mathfrak{A}$ with $\mathbf{L}^n$ via (22) by considering $\mathfrak{e}_k := e^{k-1}$, $k = 1, \dots, n$, as standard base in $\mathbf{L}^n$. Then the action ϵ of $\mathfrak{A}$ on $\mathbf{L}^n$ is given by

$$(24) \qquad\qquad \epsilon(l)\mathfrak{e}_k = le^{k-1} = e^{k-1}\sigma^{k-1}(l) = \mathfrak{e}_k \sigma^{k-1}(l), \text{ and}$$

$$\epsilon(e)\mathfrak{e}_k = ee^{k-1} = e^k = \left\{ \begin{array}{ll} \mathfrak{e}_{k+1}, & \text{if } k < n; \\ a\mathfrak{e}_1, & \text{for } k = n, \end{array} \right\}$$

That is, these formulas define an algebra homomorphism $\epsilon : \mathfrak{A} \to M_n(\mathbf{L})$. From (22) it follows that ϵ is injective and that the corresponding matrices of $\epsilon(l)$ and $\epsilon(e)$ are

$$(25) \qquad \epsilon(l) = \begin{pmatrix} l & 0 & \cdots & & & 0 \\ 0 & \sigma(l) & & & & \vdots \\ \vdots & & \sigma^2(l) & & & \\ & & & \ddots & & \\ & & & & 0 & \\ 0 & \cdots & & 0 & & \sigma^{n-1}(l) \end{pmatrix} , \quad \epsilon(e) = \begin{pmatrix} 0 & \cdots & & \cdots & & a \\ 1 & 0 & & & & \\ & 1 & \ddots & & & \\ & & & \ddots & & \\ & & & & & \\ & & & & 1 & 0 \end{pmatrix}$$

We denote by $\mathfrak{p}$ the canonical projection from $\mathfrak{A}$ onto $\mathbf{L}$, that is, $\mathfrak{p}(\sum e^k l_k) := l_0$. Note that $\mathfrak{p}$ maps self-adjoint elements of $\mathfrak{A}$ into $\mathbf{R}$. From (25) we easily derive the following

Lemma 18. $\mathfrak{p}$ *coincides with* $\mathfrak{p}_{11}(\epsilon(\cdot))$, *where* $\mathfrak{p}_{11} : M_n(\mathbf{L}) \to \mathbf{L}$ *is defined in Section 8.*

Let $\langle \cdot, \cdot \rangle_1 : \mathfrak{A} \times \mathfrak{A} \to \mathbf{L}$ be a $\mathbf{L}$-valued "inner product" on $\mathfrak{A}$ defined by

$$(26) \qquad\qquad \langle x, y \rangle_1 := \mathfrak{p}(y^* x), \ x, y \in \mathfrak{A}.$$

For $x = \sum_k e^k x_k$ and $y = \sum e^m y_m$, using (23) we compute

$$(27) \qquad\qquad \langle x, y \rangle_1 = \sum_{k,m} \mathfrak{p}(y_m^* e^{m*} e^k x_k) = \sum_k (e^{*k} e^k) y_k^* x_k.$$

Set $B = \mathrm{diag}\,(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$, where $\lambda_k := e^{*k} e^k$, $k = 1, \ldots, n-1$, $\lambda_0 = 1$ and let $P_B(\mathbf{L})$, $P_B^n(\mathbf{L})$ be as in the preceding section. As above we assume that the space of $*$-orderings of $\mathbf{R}$ is not empty, i.e. $-1 \notin P_B(\mathbf{L})$.

Definition 10. $\mathfrak{A}_+$ is the set of elements $y = y^* \in \mathfrak{A}$ such that $\mathfrak{p}(x^* y x) \in P_B(\mathbf{L})$ for all $x \in \mathfrak{A}$. The elements of $\mathfrak{A}_+$ are called positive.

Let $\langle x, y \rangle_1 = \langle Bx, y \rangle$ and the involution τ on $M_n(\mathbf{L})$ be as in Section 8.

Proposition 13. *The embedding* $\epsilon : \mathfrak{A} \to (M_n(\mathbf{L}), \tau)$ *is a* $*$-*homomorphism. Moreover,* $\epsilon(\mathfrak{A}_+) = \epsilon(\mathfrak{A}) \cap (M_n(\mathbf{L}), \tau)_+$.

Proof. The first statement follows from the computation

$$\langle \epsilon(z)x, y \rangle_1 = \mathfrak{p}(y^* z x) = \mathfrak{p}((z^* y)^* x) = \langle x, \epsilon(z^*)y \rangle_1.$$

Let $y = y^* \in \mathfrak{A}_+$ and $x \in \mathfrak{A} \simeq \mathbf{L}^n$. Then $\mathfrak{p}(x^* y x) \in P_B(\mathbf{L})$ if and only if $\langle \epsilon(y)x, x \rangle_1 \in P_B(\mathbf{L})$. Thus $\epsilon(y) \in (M_n(\mathbf{L}), \tau)_+$ if and only if $y \in \mathfrak{A}_+$. $\qquad \square$

We identify $\mathfrak{A}$ with the $*$-subalgebra $\epsilon(\mathfrak{A}) \subseteq M_n(\mathbf{L})$.

Lemma 19. $P_B(\mathbf{L}) \subseteq \mathbf{L}$ *is invariant under* σ.

Proof. It follows from (22) that $(e^* e)\sigma(l) = e^* l e$ for all $l \in \mathbf{L}$. Using this fact we get

$$\sigma(e^{*k} e^k) = e^{*(k+1)} e^{k+1} (e^* e)^{-1} = e^{*(k+1)} e^{k+1} \cdot e^* e \cdot ((e^* e)^{-1})^2 \in P_B(\mathbf{L}).$$

$\qquad\qquad \square$

It is known from the general theory that $\mathbf{L}$ is a splitting field of $\mathfrak{A}$ and $\mathfrak{A} \otimes_{\mathbf{K}} \mathbf{L} \simeq M_n(\mathbf{L})$. Further, the Galois group $Gal(\mathbf{L}/\mathbf{K})$ acts naturally by automorphisms on $M_n(\mathbf{L})$ such that $\mathfrak{A}$ is equal to the subalgebra of stable elements. Thus, the average of this action is a projection from $M_n(\mathbf{L})$ onto $\mathfrak{A}$. For $E_{mk} \otimes l \in M_n(\mathbf{L})$, $l \in \mathbf{L}$ we define

$$\mathfrak{P}(E_{mk} \otimes l) := \frac{1}{n} e^{m-k} \sigma^{-k+1}(l) = \frac{1}{n} e^{m-1} l e^{-k+1}.$$

Proposition 14. $\mathfrak{P}$ *is a well-defined* $\mathfrak{A}$-*bimodule projection from* $(M_n(\mathbf{L}), \tau)$ *onto* $\mathfrak{A}$.

Proof. We prove e.g. that $\mathfrak{P}$ preserves the involution. The other conditions follows by a similar reasoning. For $E_{mk} \otimes l \in M_n(\mathbf{L})$ we compute

$$\mathfrak{P}((E_{mk} \otimes l)^\tau) = \mathfrak{P}(B^{-1}(E_{km} \otimes l^*)B) = \mathfrak{P}(E_{km} \otimes \lambda_{k-1}^{-1}\lambda_{m-1}l^*) = \frac{1}{n}e^{k-1}\lambda_{k-1}^{-1}l^*\lambda_{m-1}e^{-m+1} =$$

$$= \frac{1}{n}e^{k-1}\left(e^{*(k-1)}e^{k-1}\right)^{-1}l^*e^{*(m-1)}e^{m-1}e^{-m+1} = \frac{1}{n}e^{*(-k+1)}l^*e^{*(m-1)} =$$

$$= \frac{1}{n}(e^{m-1}le^{-k+1})^* = (\mathfrak{P}(E_{mk} \otimes l))^*.$$

$\square$

Let $QM_{M_n(\mathbf{L}),\tau}(P_B^n(\mathbf{L}))$ denote the quadratic module in $(M_n(\mathbf{L}), \tau)$ generated by $P_B^n(\mathbf{L})$. By Proposition 11 we have $\mathfrak{A}_+ = QM_{M_n(\mathbf{L}),\tau}(P_B^n(\mathbf{L}))$. Further, let $QM_{\mathfrak{A}}(P_B(\mathbf{L}))$ be the quadratic module in $\mathfrak{A}$ generated by $P_B(\mathbf{L}) \subseteq \mathbf{L} \simeq \mathfrak{A}_0$. The next proposition is a crucial step in the proof of the Positivstellensatz below.

Proposition 15. $\mathfrak{P}$ *maps* $QM_{M_n(\mathbf{L}),\tau}(P_B^n(\mathbf{L}))$ *onto* $QM_{\mathfrak{A}}(P_B(\mathbf{L}))$.

Proof. Let $D = \mathrm{diag}\,(d_1, \ldots, d_n) \in P_B^n(\mathbf{L})$, $d_i \in P_B(\mathbf{L})$ and let $X = \sum_{ij} E_{ij} \otimes l_{ij}$ be an element from $M_n(\mathbf{L})$. The following computation is very similar to that used in the proof of Lemma 2:

$$X^\tau DX = \left(\sum_{i,j} E_{ij} \otimes l_{ij}\right)^\tau D\left(\sum_{k,m} E_{km} \otimes l_{km}\right) = B^{-1}\left(\sum_{i,j} E_{ji} \otimes l_{ij}^*\right)BD\left(\sum_{k,m} E_{km} \otimes l_{km}\right) =$$

$$(28) \quad = \left(\sum_{i,j} E_{ji} \otimes l_{ij}^*\lambda_{j-1}^{-1}\lambda_{i-1}d_i\right)\left(\sum_{k,m} E_{km} \otimes l_{km}\right) = \sum_k\left(\sum_{j,m} E_{jm} \otimes l_{kj}^*l_{km}\lambda_{j-1}^{-1}\lambda_{k-1}d_k\right) =$$

$$= \sum_k\left(\sum_j E_{kj} \otimes l_{kj}\right)^\tau (E_{kk} \otimes d_k)\left(\sum_m E_{km} \otimes l_{km}\right) = \sum_k X_k^\tau DX_k,$$

where $X_k = \sum_m E_{km} \otimes l_{km}$. We now prove that for every $k = 1, \ldots, n$

$$(29) \qquad \mathfrak{P}(X_k^\tau DX_k) = n^2 \cdot \mathfrak{P}(X_k^\tau)\mathfrak{P}(E_{kk} \otimes d_k)\mathfrak{P}(X_k) \equiv n \cdot \mathfrak{P}(X_k)^*\sigma^{1-k}(d_k)\mathfrak{P}(X_k).$$

Fix k. Using (28) we compute

$$\mathfrak{P}(X_k^\tau DX_k) = \mathfrak{P}\left(\sum_{j,m} E_{jm} \otimes l_{kj}^*l_{km}\lambda_{j-1}^{-1}\lambda_{k-1}d_k\right) = \frac{1}{n}\sum_{j,m} e^{j-m}\sigma^{-m+1}(l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1}l_{km}d_k).$$

On the other hand, using the equality $(\mathfrak{P}(Y))^* = \mathfrak{P}(Y^\tau) = \mathfrak{P}(B^{-1}Y^*B)$ we derive

$$(\mathfrak{P}(X_k))^* \mathfrak{P}(E_{kk} \otimes d_k)\mathfrak{P}(X_k) = \frac{1}{n}\mathfrak{P}\left(\sum_j E_{jk} \otimes l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1}\right) \cdot \sigma^{-k+1}(d_k) \cdot \mathfrak{P}\left(\sum_m E_{km} \otimes l_{km}\right) =$$

$$= \frac{1}{n^3}\left(\sum_j e^{j-k}\sigma^{-k+1}(l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1})\right) \cdot \sigma^{-k+1}(d_k) \cdot \left(\sum_m e^{k-m}\sigma^{-m+1}(l_{km})\right) =$$

$$= \frac{1}{n^3}\sum_{j,m} e^{j-k}\sigma^{-k+1}(l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1}d_k)e^{k-m}\sigma^{-m+1}(l_{km}) =$$

$$= \frac{1}{n^3}\sum_{j,m} e^{j-k}e^{k-m}\sigma^{-m+1}(l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1}d_k)\sigma^{-m+1}(l_{km}) = \frac{1}{n^3}\sum_{j,m} e^{j-m}\sigma^{-m+1}(l_{kj}^*\lambda_{j-1}^{-1}\lambda_{k-1}l_{km}d_k),$$

which proves (29). By Lemma 19 we obtain $\sigma^{1-k}(d_k) \in P_B(\mathbf{L})$ and the assertion follows from (28) and (29).

$\square$

Remark 2. From the proof we conlude that $\mathfrak{P}$ maps $\sum (M_n(\mathbf{L}), \tau)^2$ onto $\sum \mathfrak{A}^2$, that is, $\mathfrak{P}$ is even a strong conditional expectation.

Proposition 16. *Every element $X \in P_B(\mathbf{L}) \subseteq \mathfrak{A}$ is a non-commutative sum of squares in $\mathfrak{A}$. In the case $n = 2$ we have $P_B(\mathbf{L}) \subseteq \sum \mathfrak{A}^2$.*

Proof. The assertion follows from the equalities $\lambda_k = e^{*k}e^k$, $k = 1, \ldots, n-1$, and the fact that $P_B(\mathbf{L})$ is generated by $\lambda_1, \ldots, \lambda_{n-1}$. $\qquad \square$

Proposition 17. *If $X, Y \in \mathfrak{A}_+$ and $XY = YX$, then $XY \in \mathfrak{A}_+$. In particular, non-commutative sums of squares in $\mathfrak{A}$ belong to $\mathfrak{A}_+$.*

Proof. Combine Propositions 9 and 13. $\qquad \square$

Proposition 18. *An element $x = x^* \in \mathfrak{A}$ is in $\mathfrak{A}_+$ if and only if it belongs to $QM_{\mathfrak{A}}(P_B(\mathbf{L}))$.*

Proof. If $x \in \mathfrak{A}_+$, then we have $x \in (M_n(\mathbf{L}), \tau)_+$ by Proposition 13. From Proposition 11 we obtain $x \in QM_{M_n(\mathbf{L}),\tau}(P_B^n(\mathbf{L}))$ and from Proposition 15 we get $x \in QM_{\mathfrak{A}}(P_B(\mathbf{L}))$. On the other hand, since $P_B(\mathbf{L}) \subseteq \mathfrak{A}_+$ we obtain $\mathfrak{A}_+ \subseteq QM_{\mathfrak{A}}(P_B(\mathbf{L}))$. $\qquad \square$

Summarizing the preceding we obtain the following Posititvstellensatz.

Theorem 10. *Let $x = x^* \in \mathfrak{A}$. Then $x \in \mathfrak{A}_+$ if and only x is a non-commutative sum of squares. If $n = 2$, then $\mathfrak{A}_+ = \sum \mathfrak{A}^2$.*

Proof. Apply Propositions 16, 18 and 17. $\qquad \square$

Remarks. 1. Let $\mathbf{L} = \mathbf{R}$. An ordering p of $\mathbf{L}$ is a $*$-ordering if p contains $\mathfrak{p}(x^*x)$ for all $x \in \mathfrak{A}$, where $\mathfrak{p} : \mathfrak{A} \to \mathbf{L}$ is the canonical projection. As in the previous section one can say that $\mathfrak{A}$ *is "formally real"* if and only if there exist a $*$-ordering on $\mathbf{L}$. For cyclic algebras this seems to be the most natural analogue of the notion of formally real field. In particular, it implies that

$$(30) \qquad \qquad \sum_i x_i^* x_i = 0 \Rightarrow x_i = 0.$$

2. Let us consider the complex $*$-algebra $\mathcal{A} = \mathbb{C}\langle a, b \mid a = a^*, b = b^*, ab = e^{\mathbf{i}\frac{2\pi}{3}}ba \rangle$ and let $\mathfrak{A}$ denote the localization of $\mathcal{A}$ by its center $Z(\mathcal{A}) \simeq \mathbb{C}[a^3, b^3]$. From the representation theory of this algebra studied in [S4] it follows that $\mathcal{A}$ has a faithful $*$-representation, say π, by unbounded $*$-reprentations. Therefore, (30) is valid for $\mathcal{A}$ and hence for $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a cyclic algebra associated with the extension $\mathbb{C}(a^3, b)/\mathbb{C}(a^3, b^3)$, but the involution of $\mathfrak{A}$ does not satisfy $\mathfrak{A}_k^* = \mathfrak{A}_{-k}$. If we define $\mathcal{A}_+$ by $\mathcal{A}_+ = \{x \in \mathcal{A} : \pi(x) \geq 0\}$, it is natural to ask whether or not $x, y \in \mathcal{A}_+$ and $xy = yx$ imply that $xy \in \mathcal{A}_+$.

3. All notions and results of this section remain valid in the following more general context. Suppose that $\mathbf{L}/\mathbf{K}$ is a Galois extension with group G and $\Phi : G \times G \to \mathbf{L}^\circ$ is a 2-cocycle. Let $\mathfrak{A}$ be the crossed product algebra $(\mathbf{L}/\mathbf{K}, \Phi)$, that is, $\mathfrak{A} = \oplus_{\sigma \in G} e_\sigma \cdot \mathbf{L}$ as a right $\mathbf{L}$-linear vector space with multiplication defined by

$$\left(\sum_{\sigma \in G} e_\sigma c_\sigma \right)\left(\sum_{\rho \in G} e_\rho d_\rho \right) := \sum_{\sigma, \rho} e_{\sigma\rho} \Phi(\sigma, \rho) \rho(c_\sigma) d_\rho,$$

see [P] for more details. Then $\mathfrak{A}$ is G-graded, where $\mathfrak{A}_\sigma = e_\sigma \cdot \mathbf{L}$.

The results in [C1] (see e.g. Theorem 10 therein) show that in order to conclude that (30) is valid it is natural to require $\mathfrak{A}_\sigma^* = \mathfrak{A}_{\sigma^{-1}}$, $\sigma \in G$.

10. Examples.

Now we illustrate the results of previous two sections by examples. All three examples in this section have the following in common: $\mathcal{A}$ is a finitely generated complex $*$-algebra and $\mathcal{A}_+$ is defined by

$$\mathcal{A}_+ := \{x \in \mathcal{A}_h : \; \pi(x) \geq 0 \text{ for all finite-dimensional } * - \text{representations } \pi\}.$$

The center $Z(\mathcal{A})$ contains no zero divizors and the localization $\mathfrak{A} := \mathcal{A}(Z(\mathcal{A}))^{-1}$ of $\mathcal{A}$ by $Z(\mathcal{A})$ is a cyclic algebra of order n. Thus $\mathfrak{A}$ is a $*$-algebra over $\mathbf{K} := \mathrm{Quot}(Z(\mathcal{A}))$. In all three examples we have the equality $\mathcal{A}_+ = \mathfrak{A}_+ \cap \mathcal{A}$ which allows us to apply the results from the previous sections.

Let Rep_n be the family of all n-dimensional irreducible $*$-representations of $\mathcal{A}$ and set $n = 2$ in the first example and $n = 3$ in the second and third example. Then the set Rep_n separates the elements of $\mathcal{A}$ and an element $x \in \mathcal{A}_h$ is in $\mathcal{A}_+$ if (and only if) $\pi(x) \geq 0$ for all $\pi \in \mathrm{Rep}_n$.

10.1. **One-dimensional WCAR-algebra.** In this subsection we set

$$\mathcal{A} := \mathbb{C}\langle x, x^* | x^* x + x x^* = 1\rangle.$$

All irreducible $*$-representations of the $*$-algebra $\mathcal{A}$ are of dimensions 1 or 2, see e.g. [Sa]. Every irreducible 2-dimensional $*$-representation is of the form

$$(31) \qquad x \mapsto \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}, \; x^* \mapsto \begin{pmatrix} 0 & u \\ \overline{v} & 0 \end{pmatrix}, \; u \in \mathbb{R}, \; v \in \mathbb{C}, \; u^2 + v\overline{v} = 1.$$

In the case $u^2 = v\overline{v}$ each $*$-representation (31) is a sum of one-dimensional representations and all one-dimensional representations of $\mathcal{A}$ are obtained in this way [Sa]. This implies that $y \in \mathcal{A}_h$ is in $\mathcal{A}_+$ if and only if y is positive in all $*$-representations of the form (31). We consider u and v as commuting variables satisfying $u^2 + \overline{v}v = 1$, $u = \overline{u}$. Since the $*$-representations (31) separate the elements of $\mathcal{A}$, they define a $*$-embedding $\mathcal{A} \hookrightarrow M_2(\mathbb{C}[u,v])$. Via this embedding $\mathcal{A}$ is identified with the $*$-algebra of matrices of the form

$$(32) \qquad \begin{pmatrix} P_1 + u^2 P_2 & v P_3 + u P_4 \\ u P_3 + \overline{v} P_4 & P_1 + v\overline{v} P_2 \end{pmatrix}, \; P_i \in \mathbb{C}[uv, u\overline{v}].$$

The center $Z(\mathcal{A})$ is isomorphic to $\mathbb{C}[uv, u\overline{v}]$ and it is an integral domain. We denote by $\mathfrak{A}$ the algebra $\mathcal{A}(Z(\mathcal{A}))^{-1}$, by $\mathbf{K}$ the field $\mathrm{Quot}(Z(\mathcal{A})) = \mathbb{C}(uv, u\overline{v})$, and by $\mathbf{L}$ the field $\mathbb{C}(u^2, uv, u\overline{v})$.

The Galois extension $\mathbf{L}/\mathbf{K}$ is defined by the $\mathbf{K}$-automorphism

$$\sigma : u^2 \mapsto v\overline{v} = 1 - u^2,$$

of $\mathbf{L}$ which is of order 2. Let e be the generator x. Then $e^2 = \begin{pmatrix} uv & 0 \\ 0 & uv \end{pmatrix} = uv \cdot \mathbf{1}_{\mathfrak{A}}$. Then one easily checks that $\mathfrak{A}$ is the cyclic algebra $(\mathbf{L}/\mathbf{K}, \sigma, uv)$ which satisfies assumptions of Section 9. Let $\mathfrak{A}_+$ be as in Section 9.

Lemma 20. $\mathcal{A}_+ = \mathfrak{A}_+ \cap \mathcal{A}$.

Proof. We only sketch the proof. If we take in (32) $P_i \in \mathbf{K} = \mathbb{C}(uv, u\overline{v})$ we obtain a $*$-embedding $\mathfrak{A} \to M_2(\mathbb{C}(u, v, \overline{v}))$. We define a mapping $\phi : \mathfrak{A} \to M_2(\mathbf{L})$ by

$$(33) \qquad \begin{pmatrix} P_1 + u^2 P_2 & v P_3 + u P_4 \\ u P_3 + \overline{v} P_4 & P_1 + v\overline{v} P_2 \end{pmatrix} \mapsto \begin{pmatrix} P_1 + u^2 P_2 & uv P_3 + u^2 P_4 \\ P_3 + \frac{u\overline{v}}{u^2} P_4 & P_1 + v\overline{v} P_2 \end{pmatrix}, \; P_i \in \mathbf{L}.$$

Let $B = \mathrm{diag}\,(1, u^2) \in M_2(\mathbf{L})$ and let τ be the corresponding involution on $M_2(\mathbf{L})$, that is $\tau(X) = B^{-1} X^* B$, $X \in M_2(\mathbf{L})$. A direct computation shows that ϕ is a $*$-embedding of $\mathfrak{A}$ into $(M_2(\mathbf{L}), \tau)$. As in Section 8, let $P_B(\mathbf{L})$ be the preordering in $\mathbf{L}$ generated by u^2.

Take $y = y^* \in \mathcal{A}_+$. Let $\Delta_1, \Delta_2, \Delta_3 \in \mathbb{C}[u, v, \overline{v}]$ be the principal minors of y. Then $\Delta_i \in \mathbb{C}[u, v, \overline{v}]_+ \subseteq \mathbb{C}(u, v, \overline{v})_+$. From (32) it follows that $\Delta_i \in \mathbf{L}$. From the Krivine-Stengle theorem

(see [Ma]) we conclude that $\mathbb{C}(u, v, \overline{v})_+ = \sum \mathbb{C}(u, v, \overline{v})^2$. A simple computation shows that $\sum \mathbb{C}(u, v, \overline{v})^2 \cap \mathbf{L} = \sum \mathbf{L}^2 + u^2 \sum \mathbf{L}^2 = P_B(\mathbf{L})$. Hence $\Delta_i \in P_B(\mathbf{L})$. It follows from (33) that the corresponding leading minors of $\phi(y)$ are also equal to Δ_i. Since $\Delta_i \in P_B(\mathbf{L})$ by Lemma 14, we get $\phi(y) \in (M_2(\mathbf{L}), \tau)_+$ and Proposition 13 implies that $y \in \mathfrak{A}_+$.

A similar reasoning shows that $y \in \mathfrak{A}_+ \cap \mathcal{A}$ implies $y \in \mathcal{A}_+$. $\qquad \square$

Combining Lemma 20 with Theorem 10 we obtain the following

Theorem 11. *An element $y = y^* \in \mathcal{A}$ is in $\mathcal{A}_+$ (that is, y is positive in all finite-dimensional $*$-representations of $\mathcal{A}$) if and only if $y \in \sum \mathfrak{A}^2$, or equivalently, there exists a $c \in Z(\mathcal{A})$, $c \neq 0$, such that $c^*c \cdot y \in \sum \mathcal{A}^2$.*

Remark 3. The preceding theorem can be also obtained by using Theorem 5.4 and Corollary 5.5 in [PS], since $\mathfrak{A}$ is a quaternion algebra.

10.2. An algebra related to $\widetilde{E_6}$.

In this subsection $\mathcal{A}$ is a $*$-algebra related to the extended Dynkin diagram $\widetilde{E_6}$ (see [Me] and the references therein), that is,

$$\mathcal{A} := \mathbb{C}\langle a_1, a_2, a_3 | a_1 + a_2 + a_3 = 0, a_i^* = a_i = a_i^3 \rangle.$$

We can generate $\mathcal{A}$ by the so-called *centered* element x (see [Me]) which is defined by relations $a_i = \varepsilon^i x + \varepsilon^{-i} x$, $i = 1, 2, 3$, where $\varepsilon = e^{\mathbf{i}\frac{2\pi}{3}}$. Then we have the following lemma [Me]:

Lemma 21. *If x and x^* are taken as generators, then $\mathcal{A}$ has the form*

$$\mathcal{A} = \mathbb{C}\langle x, x^* | x^3 + x^{*3} = 0, \ x^2 x^* + x x^* x + x^* x^2 - x = 0,$$

$$x^{*2} x + x^* x x^* + x x^{*2} - x^* = 0 \rangle$$

The $*$-algebra $\mathcal{A}$ has the following family of 3-dimensional $*$-representations

$$(34) \qquad x \mapsto \begin{pmatrix} 0 & 0 & \mathbf{i}v_3 \\ v_1 & 0 & 0 \\ 0 & v_2 & 0 \end{pmatrix}, \ x^* \mapsto \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ -\mathbf{i}v_3 & 0 & 0 \end{pmatrix}$$

where $v_1, v_2, v_3 \in \mathbb{R}$ satisfying $v_1^2 + v_2^2 + v_3^2 = 1$. It can be shown that the representations (34) form a separating family, so they define a $*$-embedding $\mathcal{A} \hookrightarrow M_3(\mathbb{C}[v_1, v_2, v_3])$. In this manner $\mathcal{A}$ is identified with the $*$-algebra of matrices of the following form:

$$\begin{pmatrix} P & v_1 \cdot Q & \mathbf{i}v_3 \cdot R \\ v_1 \cdot \sigma(R) & \sigma(P) & v_2 \cdot \sigma(Q) \\ -\mathbf{i}v_3 \cdot \sigma^2(Q) & v_2 \cdot \sigma^2(R) & \sigma^2(P) \end{pmatrix}, \text{ where } P, Q, R \in \mathbb{C}[v_1^2, v_2^2, v_3^2, v_1 v_2 v_3].$$

Let σ be the automorphism of $\mathcal{B} := \mathbb{C}[v_1^2, v_2^2, v_3^2, v_1 v_2 v_3]$ defined by

$$v_1^2 \mapsto v_2^2, \ v_2^2 \mapsto v_3^2, \ v_3^2 \mapsto v_1^2, \ v_1 v_2 v_3 \mapsto v_1 v_2 v_3.$$

Then the center $Z(\mathcal{A})$ is isomorphic to the stable subalgebra of $\mathcal{B}$ under the automorphism σ and it is an integral domain. Let $\mathfrak{A}$ denote the algebra $\mathcal{A}(Z(\mathcal{A}))^{-1}$, $\mathbf{K}$ the field $\mathrm{Quot}(Z(\mathcal{A}))$, and $\mathbf{L}$ the field $\mathbb{C}(v_1^2, v_2^2, v_3^2, v_1 v_2 v_3)$.

The automorphism σ extends to $\mathbf{L}$ and defines the Galois extension $\mathbf{L}/\mathbf{K}$, which is a cyclic extension, since $\sigma^3 = \mathrm{id}$. Let e be the generator x. Then $e^3 = \mathbf{i}v_1 v_2 v_3 \cdot \mathbf{1}_\mathfrak{A}$. One checks by some direct computations that $\mathfrak{A} = \mathbf{L} + \mathbf{L}e + \mathbf{L}e^2$ is the cyclic algebra $(\mathbf{L}/\mathbf{K}, \sigma, \mathbf{i}v_1 v_2 v_3)$.

As in the previous subsection the following lemma can be proved.

Lemma 22. $\mathcal{A}_+ = \mathfrak{A}_+ \cap \mathcal{A}$.

Combining Lemma 22 and Theorem 10 it follows that an element $y = y^* \in \mathcal{A}$ is in $\mathcal{A}_+$ if and only if it is in $\sum_{nc} \mathcal{A}^2$, that is, y is a non-commutative sum of squares in $\mathfrak{A}$. In fact, the following stronger result holds.

Theorem 12. *Let $y = y^* \in \mathcal{A}$. Then $y \in \mathcal{A}_+$ if and only if $y \in \sum \mathfrak{A}^2$, or equivalently, there exists $c \in Z(\mathcal{A})$, $c \neq 0$, such that $c^*c \cdot y \in \sum \mathcal{A}^2$.*

Proof. By Proposition 18 we have $\mathfrak{A}_+ = QM_\mathfrak{A}(P_B(\mathbf{L}))$. Hence $\mathfrak{A}_+$ is the quadratic module generated by $e^*e \cdot e^{*2}e^2$ or equivalently by $e^*e \cdot ee^*$. Since $e = x$, we have $e = e^*e^2 + e^2e^* + ee^*e$. Multiplying this equation by e^* from the left and by ee^* from the right and remembering that $e^3 \in Z(\mathfrak{A})$, $ee^{*2}e = e^*e^2e^*$, we derive

$$e^*e \cdot ee^* = e^{*2}e^3e^* + e^*e^2e^*ee^* + e^*ee^*e^2e^* =$$

$$= e^{*3}e^3 + ee^{*2}e^2e^* + e^*e^2e^{*2}e = e^{*3}e^3 + (e^2e^*)^*e^2e^* + (e^{*2}e)^*e^{*2}e \in \sum \mathfrak{A}^2.$$

Thus $\mathfrak{A}_+ = QM(e^*e \cdot ee^*) = \sum \mathfrak{A}^2$. $\qquad\square$

10.3. **A counterexample to a question of Procesi and Schacher.** Procesi and Schacher [PS] asked if the denominator-free Positivstellensatz holds in central simple algebras (CSA) with involution. Recently Klep and Unger [KU] gave a nice counterexample. We now provide another counterexample which is a cyclic algebra.

Let $\mathbb{R}[x, y, z]$ be the $*$-algebra of polynomials in three real variables and $\mathcal{A}$ be the $*$-subalgebra of $M_3(\mathbb{R}[x, y, z])$ generated by the identity and the matrices

(35)
$$X = \begin{pmatrix} 0 & 0 & z \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, \quad X^* = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{pmatrix}.$$

As in the previous subsection, let $\mathfrak{A} \subseteq M_3(\mathbb{R}(x, y, z))$ denote the localization of $\mathcal{A}$ by its center. We consider $\mathfrak{A}$ with the natural involution inherited from $\mathcal{A}$.

Let $\mathbf{L} = \mathbb{R}(x^2, y^2, z^2, xyz)$ and let σ be the automorphism of $\mathbf{L}$ of order three defined by

$$\sigma(x^2) = y^2, \ \sigma(y^2) = z^2, \ \sigma(z^2) = x^2, \ \sigma(xyz) = xyz.$$

We denote by $\mathbf{K} \subset \mathbf{L}$ the stable subfield under σ. We identify $\mathbf{L}$ with a subfield of $M_3(\mathbb{R}(x, y, z))$ via the embedding

(36)
$$\mathbf{L} \ni l \mapsto \operatorname{diag}\left(l, \sigma(l), \sigma^2(l)\right).$$

By direct computations one checks that

(37)
$$\mathfrak{A} = \mathbf{L} \oplus \mathbf{L}X \oplus \mathbf{L}X^* = \mathbf{L} \oplus \mathbf{L}X \oplus \mathbf{L}X^2.$$

Lemma 23. *$\mathfrak{A}$ is isomorphic to the cyclic algebra $(\mathbf{L}/\mathbf{K}, \sigma, xyz)$.*

Proof. This follows from (37) and the following equalities

$$\operatorname{diag}\left(l, \sigma(l), \sigma^2(l)\right) \cdot X = X \cdot \operatorname{diag}\left(\sigma(l), \sigma^2(l), l\right) \in M_3(\mathbb{R}(x, y, z)),$$

$$X^* = \operatorname{diag}\left(\frac{x^2}{xyz}, \frac{y^2}{xyz}, \frac{z^2}{xyz}\right) \cdot X^2 \text{ and } X^3 = xyz \cdot I.$$

$\qquad\square$

Lemma 24. *Let $Y = (y_{ij}) \in \mathfrak{A}$ be an element of $\sum \mathfrak{A}^2$. Then $y_{11} \in \sum \mathbf{L}^2 + x^2 \cdot \sum \mathbf{L}^2 + z^2 \cdot \sum \mathbf{L}^2$.*

Proof. Take $Z = l_0 + l_1 X + l_2 X^* \in \mathfrak{A}$ and let $Y = ZZ^*$. Then $y_{11} = l_0 l_0^* + z^2 l_1 l_1^* + x^2 l_2 l_2^*$. $\qquad\square$

The following lemma occurs also in [KU]. We include an elementary proof.

Lemma 25. *Let*

(38)
$$s_0 xy = s_1 + s_2 x + s_3 y, \ s_i \in \sum \mathbb{R}(x, y)^2, \ i = 0, 1, 2, 3.$$

Then $s_i = 0$ for $i = 0, 1, 2, 3$.

Proof. After multiplying (38) by a common denominator we are reduced to the case $s_i \in \sum \mathbb{R}[x, y]^2$. Dividing both sides by a power of x^2 we can assume that x^2 does not divide all summands in s_i.

Setting $x = 0$, $y > 0$, we get $0 = s_1(0, y) + y s_3(0, y)$ which implies $s_1(0, y) = s_3(0, y) = 0$, so that s_1 and s_3 are divisible by x. Thus, each summand in s_1, s_3 is divisible by x, hence by x^2. That is, $s_1 = s_1' x^2$, $s_3 = s_3' x^2$, where $s_1', s_3' \in \sum \mathbb{R}[x, y]^2$. Cancelling x in (38) we get

$$(39) \qquad s_0 y = s_1' x + s_2 + s_3' xy.$$

Setting $x = 0$, $y < 0$, in (39) we obtain in similar manner $s_0 = x^2 s_0'$, $s_2 = x^2 s_2'$. Thus all summands in s_i, $i = 0, 1, 2, 3$, are divisible by x^2 which is a contradiction. $\qquad\square$

Lemma 26. *Suppose that we have an equality*

$$(40) \quad s_0 xy + s_1 z = s_2 + s_3 x + s_4 y + s_5 xz + s_6 yz + s_7 xyz, \ \ s_i \in \sum \mathbb{R}(x, y, z)^2, \ \ i = 0, \ldots, 7.$$

Then $s_i = 0$ for all $i = 0, \ldots, 7$.

Proof. The proof uses the same reasoning as in the proof of Lemma 25. We consider only the case $s_i \in \sum \mathbb{R}[x, y, z]^2$ and assume that z^2 does not divide all summands in s_i.

Setting $z = 0$ in (40) we get

$$(41) \qquad s_0(x, y, 0) xy = s_2(x, y, 0) + s_3(x, y, 0) x + s_4(x, y, 0) y.$$

Lemma 25 implies that s_0, s_2, s_3, s_4 are divisibe by z^2, i.e. $s_0 = s_0' z^2, s_2 = s_2' z^2, s_3 = s_3' z^2, s_4 = s_4' z^2$, where $s_i' \in \sum \mathbb{R}[x, y, z]^2$, $i = 0, 2, 3, 4$. Dividing both sides of (40) by z we obtain

$$(42) \qquad s_0' xyz + s_1 = s_2' z + s_3' zx + s_4' zy + s_5 x + s_6 y + s_7 xy, \ \ s_i, s_j' \in \sum \mathbb{R}(x, y, z)^2.$$

Seeting $z = 0$ and dividing both sides by xy in (42) we derive

$$\frac{s_1(x, y, 0)}{(xy)^2} xy = \frac{s_5(x, y, 0)}{y^2} s_5 y + \frac{s_6(x, y, 0)}{x^2} s_6 x + s_7(x, y, 0).$$

Lemma 25 implies that $s_1(x, y, 0) = s_5(x, y, 0) = s_6(x, y, 0) = s_7(x, y, 0) = 0$. Hence all elements s_i are divisible by z^2 which is a contradiction. $\qquad\square$

Then an element $(y_{ij}) \in \mathcal{A} \subset M_3(\mathbb{R}[x, y, z])$ is in $\mathcal{A}_+$ if and only if the matrix $(y_{ij}(x, y, z))$ is positive semi-definite for all $(x, y, z) \in \mathbb{R}^3$. As in the previous two examples, one can prove that $\mathcal{A}_+ = \mathcal{A} \cap \mathfrak{A}_+$, where $\mathfrak{A}_+$ is defined as in Section 9.

Proposition 19. *The matrix $Y = X^* X^2 X^* \in \mathcal{A}$ is a positive element of $\mathcal{A}$ such that $Y \notin \sum \mathfrak{A}^2$, or equivalently, there is no element $c \in Z(\mathcal{A})$, $c \neq 0$ such that $c^* c \cdot Y \in \sum \mathcal{A}^2$.*

Proof. The element Y is equal to $\operatorname{diag}(x^2 z^2, y^2 x^2, z^2 y^2)$, hence $Y \in \mathcal{A}_+$. Assume to the contrary that $Y \in \sum \mathfrak{A}^2$. Then by Lemma 24 there is an equality

$$(43) \qquad s_0 x^2 z^2 = s_1 + s_2 x^2 + s_3 z^2, \ \ \text{where} \ \ s_i \in \sum \in \mathbb{R}(x^2, y^2, z^2, xyz)^2, \ \ s_0 = 1.$$

Each element $t \in \mathbb{R}(x^2, y^2, z^2, xyz)$ can be written as $t_1 + xyz t_2$, $t_i \in \mathbb{R}(x^2, y^2, z^2)$. Hence $s_i = r_i + q_i x^2 y^2 z^2$, $r_i, q_i \in \sum \mathbb{R}(x^2, y^2, z^2)^2$. Applying this to (43) we obtain

$$r_0 x^2 z^2 + (q_0(x^2 z^2)^2) y^2 = r_1 + q_1 x^2 y^2 z^2 + r_2 x^2 + (x^4 q_2) y^2 z^2 + r_3 z^2 + (z^4 q_3) x^2 y^2.$$

Applying Lemma 26 to $\mathbb{R}(x^2, y^2, z^2)$ we get $r_i = q_i = 0$, hence $s_i = 0$ for all i. This is a contradiction, since $s_0 \neq 0$.

$\qquad\square$

11. Some problems

Let $\mathcal{W}(d)$ be the Weyl algebra, that is, $\mathcal{W}(d)$ is the unital $*$-algebra with self-adjoint generators $p_1, \ldots, p_d, q_1, \ldots, q_d$ and defining relations

$$p_j p_k = p_k p_j, \ q_j q_k = q_k q_j, \ p_j q_k = q_k p_j \ \text{for } j, k = 1 \cdots, d, j \neq k,$$
$$p_k q_k - q_k p_k = -i \ \text{for } k = 1, \ldots, d.$$

There is a distingushed faithful $*$-representation π_0 of $\mathcal{W}(d)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, called the *Schrödinger representation*, defined by

$$(\pi_0(p_k)f)(t) = \frac{\partial}{\partial t_k} f(t), \ (\pi_0(q_k)f)(t) = t_k f(t), \ \ k = 1, \cdots, d, \ f \in \mathcal{S}(\mathbb{R}^d).$$

Define

$$\mathcal{W}(d)_+ := \{x \in \mathcal{W}(d) : \langle \pi_0(x)f, f \rangle \geq 0 \ for \ f \in \mathcal{S}(\mathbb{R}^d)\}.$$

Problem 1: *Does a Positivstellensatz of type II hold for the algebra $\mathcal{W}(\mathbb{R}^d)$, that is, given $x \in \mathcal{W}(d)_+$, does there exist $c \in \mathcal{W}(d)$, $c \neq 0$, such that $cxc^* \in \sum \mathcal{W}(d)^2$? If yes, can c be chosen such that the kernel of the of operator $\overline{\pi_0(c)}$ is contained in the kernel of $\overline{\pi_0(x)}$?*

Strict Positivstellensätze and results supporting this question were proved in [S5] and in [S7].

There is a similar problem for enveloping algebras of finite dimensional Lie algebras. Let $\mathfrak{g}$ be a real Lie algebra. Then the complex universal enveloping algebra $\mathcal{E}(\mathfrak{g})$ of $\mathfrak{g}$ is a complex unital $*$-algebra with involution determined by $x^* := -x$ for $x \in \mathfrak{g}$. Let G denote the connected simply connected Lie group which $\mathfrak{g}$ as its Lie algebra and let $\widehat{G}$ be the set of unitary equivalence classes of irreducible unitary representations of G. For each $U \in \widehat{G}$ there is an associated $*$-representation dU of $\mathcal{E}(\mathfrak{g})$ with domain $\mathcal{D}^\infty(U)$, see [S1], Chapter 10. Now let

$$\mathcal{E}(\mathfrak{g})_+ := \left\{ x \in \mathcal{E}(\mathfrak{g}) : \langle dU(x)f, f \rangle \geq 0 \ \text{for } f \in \mathcal{D}^\infty(U), \ U \in \widehat{G} \right\}.$$

The counterpart of Problem 1 for enveloping algebras is the following

Problem 2: *Is a Positivstellensatz of type II true for $\mathcal{E}(\mathfrak{g})$, that is, given $x \in \mathcal{E}(\mathfrak{g})_+$, does there exist an element $c \in \mathcal{E}(\mathfrak{g})$, $c \neq 0$, such that $cxc^* \in \sum \mathcal{E}(\mathfrak{g})^2$?*

In both types III and IV commuting positive elements occur. While the product of two commuting positive bounded operators on a Hilbert space is always positive (Lemma 1), there are examples of commuting positive symmetric operators on a unitary space for which the product is no longer positive. It seems to be unknown whether or not the latter can happen in the Schrödinger representation of the Weyl algebra.

Problem 3: *Suppose that $a, b \in \mathcal{W}(d)_+$ and $ab = ba$. Is it true that $ab \in \mathcal{W}(d)_+$?*

An affirmative answer would imply that all elements of the minimal non-commutative pre-ordering in the Weyl algebra are indeed positive elements.

Let $\mathcal{A}$ be a unital $*$-algebra and $\mathcal{R}$ a separating family of $*$-representations of $\mathcal{A}$. If π is a $*$-representation of $\mathcal{A}$ on $\mathcal{D}$, there is a unique $*$-representation π_n of the matrix $*$-algebra $M_n(\mathcal{A})$ on $\mathcal{D}_n = \mathcal{D} \oplus \cdots \oplus \mathcal{D}$ (n times) defined by $\pi_n((a_{kl})) := (\pi(a_{kl}))$.

Let $\mathcal{A}_+ := \{a \in \mathcal{A} : \pi(a) \geq 0 \ \text{for } \pi \in \mathcal{R}\}$ and $M_n(\mathcal{A})_+ := \{A \in \mathcal{A} : \pi_n(A) \geq 0 \ \text{for } \ \pi \in \mathcal{R}\}$.

Problem 4: *Suppose that a Positivstellensatz of type I hold for $\mathcal{A}$. Does it hold also for the matrix algebra $M_n(\mathcal{A})$?*

In particular, Problem 4 is open and important when $\mathcal{A}$ is the commutative real $*$-algebra of all polynomials on the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and $\mathcal{R}$ is the set of all point evaluations $\pi_t(p) = p(t)$, $t \in S^2$. It follows from results in [Sd] that each nonnegative polynomial on S^2 is a sum of squares of polynomials, that is, a Positivstellensatz of type I is

valid for $\mathcal{A}$. By the results of Section 4 there is a Positivstellensatz of type II for the matrix algebra $M_n(\mathcal{A})$. The question is whether or not a *denominator free* Positivstellensatz holds for $M_n(\mathcal{A})$.

An affirmative answer to Problem 4 in the latter case would yield a number of other interesting results. First, using a similar conditional expectation as in Section 7 it would follow that *a self-adjoint matrix polynomial $F(x,y) \in M_n(\mathbb{C}[x,y])$ is positive semi-definite on the unit disc $\{x^2 + y^2 \leq 1\}$ if and only if $F \in \sum M_n(\mathbb{C}[x,y])^2 + (1 - x^2 - y^2) \sum M_n(\mathbb{C}[x,y])^2$.*

Secondly, it would imply that a Positivstellensatz of type I holds for a number of algebras which can be embedded into $M_n(\mathbb{C}[S^2])$. For example, consider the following "non-sommutative sphere"

$$\mathcal{A} = \mathbb{C}\langle x_1, x_2, x_3 | x_1^2 + x_2^2 + x_3^2 = I, \ x_i x_j = -x_j x_i, \ x_i^* = x_i, \ i = 1, 2, 3\rangle.$$

It follows from the description of irreducible representations of $\mathcal{A}$ (see e.g. [OS], p.110) that $\mathcal{A}$ can be embedded into $M_2(\mathbb{C}[S^2])$ such that $\mathcal{A}_+ \subseteq M_2(\mathbb{C}[S^2])_+$. Further, there exists a strong conditional expectation $\mathfrak{P} : M_2(\mathbb{C}[S^2]) \to \mathcal{A}$. Hence an affirmative answer to Problem 4 for the algebra of polynomials on the sphere S^2 would give a Positivstellensatz of type I for $\mathcal{A}$.

Let G be a discrete group and let $\mathbb{C}[G]_+$ denote the set of elements in the group algebra which are positive in all $*$-representations.

Problem 5: *For which discrete groups G a Positivstellensatz of type I holds, that is, when is $\mathbb{C}[G]_+ = \sum \mathbb{C}[G]^2$?*

First let $G = \mathbb{Z}^n$. Then $\mathbb{C}[\mathbb{Z}^n]$ is isomorphic to the coordinate ring of the n-torus $\mathbb{T}^n$. or equivalently, to the $*$-algebra of trigonometric polynomials in n variables. Therefore the answer is affirmative for $n = 1$ by the classical Fejer-Riesz theorem and for $n = 2$ by the results in [Sd] and it is negative for $n \geq 3$ ([Ru], see e.g. [Dr]). Using these facts it is easily shown that for an abelian group G the equality $\mathbb{C}[G]_+ = \sum \mathbb{C}[G]^2$ holds if and only if the torsion-free component of G is either $\mathbb{Z}$ or $\mathbb{Z}^2$.

Using similar techniques as in to [HMP] one can show $\mathbb{C}[G]_+ = \sum \mathbb{C}[G]^2$ if G is a free group.

Let $H \subseteq G$ be a subgroup of a finite index $[G : H] = n$. Then there is a natural construction of an embedding $\mathbb{C}[G] \subseteq M_n(\mathbb{C}[H])$ and of a strong conditional expectation $\mathfrak{P} : M_n(\mathbb{C}[G]) \to \mathbb{C}[H]$ similar to the one in Section 5. Recall that G is called virtually free (resp. cyclic) if G contains a free (resp. cyclic) group of a finite index. In view of the Problem 4 we conjecture that $\mathbb{C}[G]_+ = \sum \mathbb{C}[G]^2$ *holds for virtually free and virtually cyclic groups of rank 2.* In the case when $\mathbb{Z} \subseteq G$ and $[G : \mathbb{Z}] < \infty$ the equality $\mathbb{C}[G]_+ = \sum \mathbb{C}[G]^2$ can be obtained from Proposition 6.

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